CLASS: BBA-5th Semester

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OPERATION RESEARCH

UNIT-3

Introduction to Sequencing:
Sequencing Problems,
Solution to Sequencing Problem - Processing n-jobs through one machine, Processing n-jobs through two machines,
Processing n-jobs through Three Machines, Processing two through m- machines, processing n-jobs through m-machines.

Network Models:
PERT AND CPM
Introduction
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INTRODUCTION

The short-term schedules show an optimal order (sequence) and time in which jobs are processed as well as show timetables for jobs, equipment, people, materials, facilities and all other resources that are needed to support the production plan. The schedules should use resources efficiently to give low costs and high utilisations. Other purpose of scheduling are, minimising customers wait time for a product, meeting promised delivery dates, keeping stock levels low, giving preferred working pattern, minimising waiting time of patients in a hospital for different types of tests and so on.

The general scheduling or sequencing problem may be described as: Let there be n jobs to be performed one at a time on each of m machines. The sequence (order) of the machines in which each job should be performed is given. The actual or expected time required by the jobs on each of the machines is also given. The general sequencing problem, therefore, is to find the sequence out of $(n!)^m$ possible sequences which minimise the total elapsed time between the start of the job in the first machine and the completion of the last job on the last machine.

In particular, if n = 3 and m= 3, then total number of possible sequences will be $(3!)^3 = 216$. Theoretically, it may be possible to find optimum sequence but it will require a big computational time. Thus, one should adopt sequencing technique.
To find optimum sequence we first calculate the total elapsed time for each of the possible sequences. As stated earlier, even if values of m and n are very small, it is difficult to get the desired sequence with total minimum elapsed time. However, due to certain rules designed by Johnson, the task of determining an optimum sequence has become quite easy.

NOTATIONS, TERMINOLOGY AND ASSUMPTIONS

Notations

t_{ij} = Processing time (time required) for job i on machine j.
T = Total elapsed time for processing all the jobs. This includes idle time, if any.
I_{ij} = Idle time on machine j from the end of job (j - 1) to the start of job i.

Terminology

- **Number of Machines:** The number of machines refer to the number of service facilities through which a job must pass before it is assumed to be completed.
- **Processing Time:** It is the time required by a job on each machine.
- **Processing Order:** It refers to the order (sequence) in which machines are required for completing the job.
- **Idle Time on a Machine:** It is the time for which a machine does not have a job to process, i.e idle time from the end of job (i-1) to the start of job i.
- **Total Elapsed Time:** It is the time interval between starting the first job and completing the last job including the idle time (if any) in a particular order by the given set of machines.
- **No Passing Rule:** It refers to the rule of maintaining the order in which jobs are to be processed on given machines. For example, if n jobs are to be processed on two machines M_1 and M_2 in the order M_1 M_2, then each job should go to machine M_1 first and then to M_2.

Assumptions

1. The processing time on different machines are exactly known and are independent of the order of the jobs in which they are to be processed.
2. The time taken by the job in moving from one machine to another is negligible.
3. Once a job has begun on a machine, it must be completed before another job can begin on the same machine.
4. All jobs are known and are ready for processing before the period under consideration begins.
5. Only one job can be processed on a given machine at a time.
6. Machines to be used are of different types.
7. The order of completion of jobs are independent of the sequence of jobs.

PROCESSING n JOBS THROUGH TWO MACHINES

Let there be n jobs, each of which is to be processed through two machines, M_1 and M_2 in the order M_1 M_2, i.e. each job has to be passed through the same sequence of operations. In other words, a job is assigned on machine M_1 first and after it has been completely processed on machine M_1, it is assigned to machine M_2. If the machine M_2 is not free at the moment for processing the same job, then the job has to wait in waiting line for its turn on machine M_2, i.e. passing is not allowed.

Since passing is not allowed, therefore, machine M_1 will remain busy in processing all the n jobs one-by-one which machine M_2 may remain idle time of the second machine. This can be achieved only by determining sequence of n jobs which are to be processed on two machines M_1 and M_2. The procedure suggested by Johnson for determining the optimal sequence can be summarised as follows:

The Algorithm

Step 1 List the jobs along with their processing times on each machine in a table as shown below:

Processing Time on Machine	Job Number			
	1	2	3	n
M_1	t_{11}	t_{12}	t_{13}	t_{1n}
M_2	t_{21}	t_{22}	t_{23}	t_{2n}

Step 2 Examine the columns for processing times on machines M_1 and M_2, and find the smallest processing time in each column, i.e find out, min. (t_{1j}, t_{2j}) for all j.

Step 3(a) If the smallest processing time is on machine M_1, then schedule the job as early as possible without moving jobs already schedules, i.e place the job in the first available position in the sequence. If the processing time is on machine M_2, then schedule the job as late as possible without moving any jobs already scheduled, i.e. place the job in the last available position in the sequence.

b. If there is a tie in selecting the minimum of all the processing times, then there may be three situations:

a. Minimum among all processing times is same for the machine i.e. min (t_{1j}, t_{2j}) = t_{2k} = t_{2r}, then process the kth job first and the rth job last.
b. If the tie for the minimum occurs among processing times t_{1j} on machine M_1 only, then select the job corresponding to the smallest job subscript first.
c. If the tie for the minimum occurs among processing times t_{2j} on machine M_2, then select the job corresponding to the largest job corresponding to the largest job subscript last.

Step 4 Remove the assigned jobs from the table. If the table is empty, stop and go to Step 5. Otherwise, got to Step 2.

Step 5 Calculate idle time for machines M_1 and M_2:
a. Idle time for machine M_1 = (Total Elapsed Time) - (Time when the last job in a sequence finishes on machine M_1)
b. Idle time for machine M_2 = Time at which the first job in a sequence finishes on machine M_1 + $\sum_{j=2}^{n}$ {(Time when the jth job in a sequence starts on machine M_2) - (Time when the (j - 1)th job in a sequence finishes on machine M_2)}.

Step 6 The total elapsed time to process all jobs through two machines is given by
Total Elapsed time = Time when the nth job in a sequence finishes on Machine M_2.
= $\sum_{j=1}^{n} M2j$ + $\sum_{j=1}^{n} I2j$

where M_{2j} = Time required for processing jth job on machine M_2.

I_{2j} = Time for which machine M_2 remains idle after processing (j - 1)th job and before starting work in jth job.

EXAMPLE 1

Find the sequence that minimises the total elapsed time required to complete the following tasks on two machines:

Task	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**
Machine I	**2**	**5**	**4**	**9**	**6**	**8**	**7**	**5**	**4**
Machine II	**6**	**8**	**7**	**4**	**3**	**9**	**3**	**8**	**11**

Solution:

The smallest processing time between the two machines is 2 which corresponds to task A on Machine I. Thus, task A is scheduled as early as possible to give the sequence as shown below:

A								

After the task A has been set for processing first, we are left with 8 tasks and their processing times as given below:

Task	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**
Machine I	**5**	**4**	**9**	**6**	**8**	**7**	**5**	**4**
Machine II	**8**	**7**	**4**	**3**	**9**	**3**	**8**	**11**

The minimum processing time in this reduced problem is 3 which corresponds to task E and G both on machine II. Since the corresponding processing time of task E on machine I is less than the corresponding processing time of task G on machine I, therefore, task E will be scheduled in the last and task G shall be scheduled before it. Tasks E and G will not be considered further. Thus, current partial sequence of scheduling tasks becomes:

A							G	E

A set of processing times now gets reduced to:

Task	**B**	**C**	**D**	**F**	**H**	**I**
Machine I	**5**	**4**	**9**	**8**	**5**	**4**
Machine II	**8**	**7**	**4**	**9**	**8**	**11**

The smallest processing time in this reduced problem is 4, which corresponds to task C and I on machine I and to task D on machine II. Thus task C will be placed in the second sequence cell and task I in the third sequence cell and task D in the sequence cell before task G. The entries of the partial sequence are now:

A	C	I				D	G	E

The set of processing time now gets reduced as follows:

Task	**B**	**F**	**H**
Machine I	**5**	**8**	**5**
Machine II	**8**	**9**	**8**

the smallest processing time in this reduced problem is 5, which corresponds to tasks B and H both on machine I. Since the corresponding processing times of B and h on machine II is same, therefore, either of these two tasks can be placed in fourth and fifth sequence cells. Thus, it indicates an alternative optimal sequence. the optimal sequences are, therefore, given below:

A	C	I	B	H	F	D	G	E

A	C	I	H	B	F	D	G	E

The minimum elapsed time for machines I and II is calculated as shown in Table 1.

Task Sequence	Machine I	Machine II

	Time In	Time Out	Time In	Time Out
A	0	2	2	8
C	2	6	8	15
I	6	10	15	26
B	10	15	26	34
H	15	20	34	42
F	20	28	42	51
D	28	37	51	55
G	37	44	55	58
E	44	50	58	61

In table 1, the minimum elapsed time, i.e time from start of task A to completion of last task E is 61 hours. During this time the machine I remains idle for 61 - 50 = 11 hours. The idle time for machine II is equal to the time at which the first task A in the sequence finishes on machine I plus the last task E in the sequence starts on machine II minus the last but one task G finishes on machine II, i.e 2 + 58 - 58 = 2 hours.

PROCESSING n JOBS THROUGH THREE MACHINES

Johnson provides an extension of his procedure to the case in which there are three instead of two machines. Each job is to be processed through three machines M_1, M_2 and $M_{3.}$ The list of jobs with their processing times is given below. An optimal solution to this problem can be obtained if either or both of the following conditions hold good.

Processing time on Machine	Job Number			
	1	2	3	4
M_1	t_{11}	t_{12}	t_{13}	t_{1n}
M_2	t_{21}	t_{22}	t_{23}	t_{2n}
M_3	t_{31}	t_{32}	t_{33}	t_{3n}

1. The minimum processing time on machine M_1 is at least as great as the maximum processing time on machine $M_{2,}$ that is,

$$\min t_{1j} \geq \max t_{1j,} \qquad j = 1, 2, 3, ...n$$

2. The minimum processing time on machine M_3 is at least as great as the maximum processing time on machine M_2, that is

$$\min {}_{t3j} \geq \max {}_{t2j,} \qquad j = 1, 2, 3, ...n$$

If either or both the above conditions hold good, then the steps of the algorithm can be summarised in the following steps:

THE ALGORITHM

Step 1: Examine processing times of given jobs on all three machines and if either one or both the above conditions hold, then go to step 2, otherwise the algorithm fails.

Step 2: Introduce two fictitious machines, say G and H with corresponding processing times given by

i. $t_{Gj} = t_{1j} + t_{2j,}\ \ j = 1, 2, 3, ... n.$

that is, processing time on machine G is the sum of the processing times on machines M_1 and $M_{2,}$ and

ii. $t_{Hj} = t_{2j} + t_{3j,}\ \ j = 1, 2, 3, ... n.$

that is, processing time on machine H is the sum of the processing times on machines M_2 and

M_3.

Step 3: Determine the optimal sequence of jobs for this n-job, two machine equivalent sequencing problem with the prescribed ordering GH in the same way as discussed earlier.

EXAMPLE

Find the sequence that minimises the total time required in performing the following jobs on three machines in the order ABC. Processing times (in hours) are given in the following table:

Job	1	2	3	4	5
Machine A	8	10	6	7	11
Machine B	5	6	2	3	4
Machine C	4	9	8	6	5

Solution: Here, min $(t_{Aj}) = 6$; Min $(t_{Cj}) = 4$; max $(t_{Bj}) = 6$. Since min $(t_{Aj}) \geq (t_{Bj})$ for all j is satisfied, the given problem can be converted into a problem of 5 jobs and two machines. The processing time on two fictitious machines G and H can be determined by the following relationships:

$t_{Gj} = t_{Aj} + t_{Bj,}$ j = 1, 2, 3, n.

and $t_{Hj} = t_{Bj} + t_{Cj,}$ j = 1, 2, 3, n.

The processing times for the new problem are given below:

Job	1	2	3	4	5
Machine G Machine H	8 + 5 = 13 5 + 4 = 9	10 + 6 = 16 6 + 9 = 15	6 + 2 = 8 2 + 8 = 10	7 + 3 = 10 3 + 6 = 9	11 + 4 = 15 4 + 5 = 9

When the algorithm described for n jobs on two machines is applied to this problem, the optimal sequence so obtained is given by

3	2	5	1	4

The total minimum elapsed time is given in Table 1:

Job Sequence	Machine A		Machine B		Machine C	
	Time In	Time Out	Time In	Time Out	Time In	Time Out
3	0	6	6	8	8	16
2	6	16	16	22	22	31
5	16	27	27	31	31	36
1	27	35	35	40	40	44
4	35	42	42	45	45	51

Table 1 indicates that the minimum total elapsed time is 51 hours. The idle time for machines A, B and C is 9 (=51 - 42) hours, 6 (= 51 - 45) hours and 9 (= 8 - 0) + (45 - 44) hours, respectively.

PROCESSING n JOBS THROUGH m MACHINES

Let there be n jobs, each of which is to be processed through m machines, say M_1, $M_{2,}$.........M_m in the order M_1, $M_{2,}$........ M_m. The optimal solution to this problem can be obtained if either or both of the following conditions hold good.

(a) Min $\{t_{1j}\} \geq$ Max $\{t_{1j}\}$; j = 2, 3, ... m - 1

and or (b) Min $\{t_{mj}\} \geq$ Max $\{t_{ij}\}$; j = 2, 3, ... m - 1

that is, the minimum processing time on machines M_1 and M_m is as great as the maximum processing time on any of the remaining (m - 1) machines.

If either or both these conditions hold good, then the steps of the algorithm can be summarised in the following steps:

Step 1: Find, Min $\{t_{1j}\}$, Min $\{t_{mj}\}$ and max $\{t_{ij}\}$and verify above conditions. If either or both the conditions mentioned above hold, then go to step 2. Otherwise the algorithm fails.

Step 2: Convert m-machine problem into 2-machine problem by introducing two fictitious machines, say

(i) $t_{Gj} = t_{1j} + t_{2j} + t_{3j} + \ldots\ldots + t_{m-1j}$ j = 1, 2, 3,..... n.

i.e. processing time of n-jobs on machine G is the sum of the processing times on Machines M_1, M_2 ....M_{m-1j}

(ii) $t_{Hj} = t_{2j} + t_{3j} + t_{4j} + \ldots\ldots + t_{mj}$ j = 1, 2, 3,......n.

i.e. processing time of n-jobs on machine H is the sum of the processing times on Machines M_1, M_2 M_{mj}.

Step 3: The new processing times so obtained can now be used for solving n-job, two machines equivalent sequencing problem with the prescribed ordering HG in the same way as

$t_{2j} + t_{3j} + ... + t_{m-1j} = k$ (constant)

for all j = 1, 2, 3, ... m - 1, then the optimal sequence can be obtained for n-jobs and two machines M_1 and M_m in the order M_1 M_m as usual.

2. If $t_{1j} = t_{mj}$ and $t_{Gj} = t_{Hj}$, for all j = 1, 2, 3, n, then total number of optimal sequences will be n and total minimum elapsed time in these cases would also be the same.

3. The method described above for solving n-jobs and m-machines sequencing problem is not a general method. It is applicable only to certain problems where the minimum cost (or time) of processing the jobs through first and/or last machine is more than or equal to the cost (or time) of processing the jobs through remaining machines.

EXAMPLES

1. Find an optimal sequence for the following sequencing problems of four jobs and five machines when passing is not allowed of which processing time (in hours) is given below:

Job	Machines				
	M_1	M_2	M_3	M_4	M_5
A	7	5	2	3	9
B	6	6	4	5	10
C	5	4	5	6	8
D	8	3	3	2	6

Also find the total elapsed time.

Solution: Here,

Min $(t_{M1,j}) = 5 = t_{M1,C}$

Min $(t_{M5,j}) = 6 = t_{M5,D}$

and Max $\{t_{M2,j}, t_{M3,j}, t_{M4,j}\} = \{6, 5, 6\}$ respectively.

Since the condition of Min $(t_{M5,j}) \geq$ Max $\{t_{M2,j}, t_{M3,j}, t_{M4,j}\}$ is satisfied, therefore the given problem can be converted into a four jobs and two machines problem as G and H. The processing times of four jobs denoted by t_{Gj} and t_{Hj} on G and H, respectively are as follows:

Job	A	B	C	D
Machine G	17	21	20	16
Machine H	19	25	23	14

where $t_{Gj} = \sum_{i=1}^{m-1} tij$ and $t_{Hj} = \sum_{i=2}^{m} tij$.

Now using the optimal sequence algorithm, the following optimal sequence can be obtained.

A	C	B	D

The total elapsed time corresponding to the optimal sequence can be calculated as shown in Table 1, using the individual processing times given in the original problem.

Table 1 shows that the minimum total elapsed time is 51 hours. The idle time for machines M_1, M_2, M_3, M_4 and M_5 is 25, 33, 37 and 18 hrs respectively.

Table 1 Minimum Elapsed Time

Job Sequence	Machine				
	M_1	M_2	M_3	M_4	M_5
A	0 - 7	7 - 12	12 - 14	14 - 17	16 - 26
B	7 - 12	12 - 18	16- 21	21 - 27	27 - 35

C D	12 - 18 18 - 26	18 - 24 26 - 29	24 - 28 29 - 32	28 - 33 33 - 35	35 - 45 45 - 51

2. Solve the following sequencing problem giving an optimal solution when passing is not allowed.

Machine	Job				
	A	B	C	D	E
M_1	11	13	9	16	17
M_2	4	3	5	2	6
M_3	6	7	5	8	4
M_4	15	8	13	9	11

Solution: From the data of the problem it is observed that

Min ($t_{M1, j}$) = 9 = $t_{M1, C}$

Min ($t_{M4, j}$) = 8 = $t_{M4, B}$

and Max { $t_{M2, j}$} = 6 = , $t_{M2, E;}$ Max { $t_{M3, j}$} = 8 = , $t_{M2, D}$.

Since both the conditions

Min ($t_{M1, j}$) ≥ Max { $t_{M2, j;}$ $t_{M3, j}$} ; j = 1, 2, ..., 5

are satisfied, therefore given problem can be converted into a 5-jobs and 2-machine problem as G and H.

Further, it may be noted that, $t_{M2, j}$ + $t_{M3, j}$ = 10 (a fixed constant) for all j (j =1, 2, ..., 5).

Thus the given problem is reduced to a problem of solving 5-jobs through 2-machines M_1 and M_4 in the order M_1 M_4. This means machines M_2 and M_4 will have no effect on the optimality of the sequences.

The processing times of 5 jobs on machine M_1 and M_4 is as follows:

Job	A	B	C	D	E
Machine M_1	11	13	9	16	17
Machine M_4	15	8	13	9	11

Now using the algorithm described earlier, the optimal sequence so obtained as follows:

C	A	E	D	B

The total elapsed time corresponding to the optimal sequence is 83 hours as shown in table 1, using the individual processing times given in the original problem:

Table 1 Minimum Total Elapsed Time

Job Sequence	Machine				
	M_1	M_2	M_3	M_4	
C	0 - 9	9 - 14	14 - 19	19 - 32	
A	9 - 20	20 - 24	24 - 30	32 - 45	
E	29 - 36	36 - 42	42 - 46	46 - 57	
D	36 - 52	52 - 54	54 - 62	62 - 71	
B	52 - 65	65 - 68	68 - 75	75 - 83	

PROCESSING TWO JOBS THROUGH m MACHINES

Let there be two jobs A and B each of which is to processed on m machines say M_1, $M_{2,}$, M_m in two different orders. The technological ordering of each of the two jobs through m machines is known in advance. Such ordering may not be same for both the jobs. The exact or expected processing times on the given machines are known. Each machine can perform only one job at a time. The objective is to determine an optimal sequence of processing the jobs so as to minimise total elapsed time.

The optimal sequence in this case can be obtained by using graph. The procedure can be illustrated by taking examples.

Example 1: Use the graphical method to minimise the time needed to process the following jobs on the machines shown, i.e. each machine find the job which should be done first. Also calculate the total elapsed time to complete both jobs.

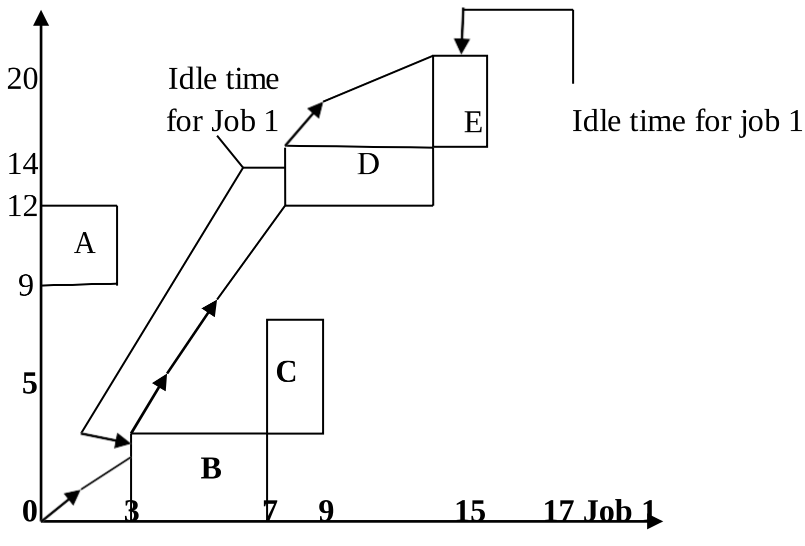
	Machine				
Job 1 {Sequence:	A	B	C	D	E
Time (hrs)	3	4	2	6	2

	Machine				
Job 2 {Sequence:	A	B	C	D	E
Time (hrs)	5	4	3	2	6

Solution

The solution procedure for solving the above problem can be summarised in the following steps:

1. Draw the set of axes at right angle to each other where x-axis represents the processing time of job 1 on different machines while job 2 remains idle and y-axis represents processing time of job 2 while job 2 remain idle.
2. Mark the processing times for jobs 1 and 2 on x-axis and y-axis, respectively according to the given order of machines as shown in the figure:



Graphical solution of 2-Jobs and m-Machines Sequencing Problem

For example, machine A takes 3 hours for job 1 and 3 hours for job 2. Construct the rectangle for machine A as shown in above Figure. Similarly, construct other rectangles for machines B, C, D and E.

3. Construct various blocks starting from the origin by pairing the same machine until a point marked 'finished' is obtained.

4. Draw a line starting from origin to the point marked 'finish' by moving horizontally, vertically and diagonally along a line which makes an angle of 45° with the horizontal axis. Moving horizontally along this line indicates that first job is under process while second job is idle. Similarly, moving vertically along this line indicates that the second job is under process while first job is idle. The diagonal movement along this line shows that both the jobs are under process simultaneously.

Since simultaneous processing of both jobs on a machine is not possible, therefore, diagonal movement is not allowed. In other words, diagonal movement through rectangle areas is not allowed.

5. An optimal path is one that minimises the idle time for both the jobs. Thus, we must choose the path on which diagonal movement is maximum as shown in the above figure.

6. Total elapsed time is obtained by adding the idle time for either job to the processing time for that job. In this example, the idle time for the chosen path is found to be 5 hrs and 2 hrs for jobs 1 and 2, respectively. The total elapsed time is calculated as follows:

Elapsed time, Job 1 = Processing time of Job 1 + Idle time for Job 1

= 17 + (2 + 3) = 22 hrs.

Elapsed time, Job 2 = Processing time of Job 2 + Idle time for Job 2

= 22 + (17-15) = 24 hours.

EXERCISES

1. Explain the four elements that characterise a sequencing problem.
2. Explain the principal assumptions made while dealing with sequencing problems.
3. Give Johnson's procedure for determining an optimal sequence for processing n items on two machines. Give justification of the rule used in the procedure.
4. What is no passing rule in a sequencing algorithm? Explain the principal assumptions made while dealing with sequencing problems.
5. Give three different examples of sequencing problems from your daily life.
6. We have five jobs, each of which must be processed on the two machines A and B in the order AB. Processing times in hours are given in the table below:

Job	1	2	3	4	5
Machine A	5	1	9	3	10
Machine B	2	6	7	8	4

7. We have 5 jobs each of which must go through the machines A, B and C in the order ABC. Processing times (in hours) is as follows:

Job	1	2	3	4	5
Machine A	5	7	6	9	5
Machine B	2	1	4	5	3
Machine C	3	7	5	6	7

8. What do you understand by the following terms in the context of sequence of jobs:
1. Job arrival pattern 2. Number of machines 3. The flow pattern in the shop
4. the criteria of evaluating the performance of a schedule.

9. Find an optimal sequence for the following sequencing problem of four jobs and five machines (when passing is not allowed) of which processing time (in hrs) is as follows:

Job	1	2	3	4
Machine M_1	6	5	4	7
Machine M_2	4	5	3	2
Machine M_3	1	3	4	2
Machine M_4	2	4	5	1
Machine M_5	8	9	7	5

Also find the total elapsed time.

10. Two jobs are to be processed on four machines A, B, C and D. The technological order for these jobs on machines is as follows:

Job 1: A B C D Job 2: D B A C

Processing times are given in the following table:

	Machines			
	A	B	C	D
Job 1	4	6	7	3
Job 2	4	7	5	8

Find the optimal sequence of jobs on each of the machines.

Network Models:

A Project such as setting up of a new milk plant, research and development in an organization, development of a new milk product, marketing of a product etc. is a combination of interrelated activities (tasks) which must be executed in a certain order before the entire task can be completed. The activities are interrelated in a logical sequence in such a way that same activities can not start until some others are completed. An activity in a project usually viewed as job requiring resources for its completion. The objectives of project management can be described in terms of a successful project which has been finished on time, within the budgeted cost and to technical specifications and to the satisfaction level of end users. Normally for any project, one may be interested in answering questions such as

i) What will be the expected time of project completion?
ii) What is the effect of delay of any activity on the overall completion of project?
iii) How to reduce the time to perform certain activities in case of availability of additional funds?
iv) What is the probability of completion of project in time?

The OR techniques used for planning, scheduling and controlling large and complex projects are often referred to as network analysis. A network is a graphical representation consisting of certain configuration of arrows and nodes for showing the logical sequence of various tasks to be performed to achieve the project objectives. Around five decades ago the planning tool was *Gantt bar chart* which specifies start and finish time for each activity on a horizontal time scale. The disadvantage is that there is no interdependency among the many activities which control the progress of the project. Now-a-days we use a technical tool for planning, scheduling and controlling stages of the projects known as Critical Path Method (CPM) and Project Evaluation & Review Technique (PERT). The techniques of PERT and CPM prove extremely valuable in assisting the mangers in handling such projects and thus discharging their project management responsibilities both at planning and controlling stages of the projects. Commonly used project management techniques are:

a) Critical Path Method (CPM) and
b) Project Evaluation and Review Technique (PERT)

18.2 Historical Development

CPM/PERT or Network Analysis as the technique is sometimes called, developed along two parallel streams, one industrial and the other military. CPM was developed in 1957 by J. E. Kelly of Remington Rand and M. R. Walker of E. I. Du Pont de Nemours & Co. PERT was devised in 1958 for the POLARIS missile program by the Program Evaluation Branch of the Special Projects office of the U. S. Navy, helped by the Lockheed Missile Systems division and the Consultant firm of Booz-Allen & Hamilton.

Both are basically time oriented methods laid to determination of a time schedule for project. The major difference between these two techniques is that **PERT** is a **Probabilistic** approach for the determination of time estimates of different activities not exactly known to us**.** In the case of **CPM,** different estimates are known as they are **deterministic** in nature. But now a days both these techniques are used for one purpose. Initially the PERT technique was applied to research and development projects while the CPM was used towards construction projects.

18.3 Methodology in CPM/PERT Technique

The methodology involved in network scheduling by CPM/PERT for any project consists of the following four stages:

18.3.1 Planning

It is started by splitting the total project into small projects. The smaller projects are further divided into different activities and are analyzed by a department or section. The relationship of each activity with respect to other activities are defined and established.

18.3.2 Scheduling

The objective of scheduling is to give the earliest and the latest allowable start and finish time of each activity as well as its relationship with other activities in the project. The schedule must pinpoint the critical path i.e. time activities which require special attention if the project is to be completed in time.

18.3.3 Allocation of resources

Allocation of resources is performed to achieve the desired objective. Resource is a physical variable such as labour, finance, space, equipment etc. which will impose a limitation for completion of a project.

18.3.4 Controlling

The final phase in the project management is controlling. After making the network plan and identification of the Critical path, the project is controlled by checking progress against the schedule, assigning and scheduling manpower and equipment and analyzing the effects of delays. This is done by progress report from time to time and updating the network continuously. Arrow diagram and time charts are used for making periodic progress reports.

18.4 Basic Terminology used in Network Analysis

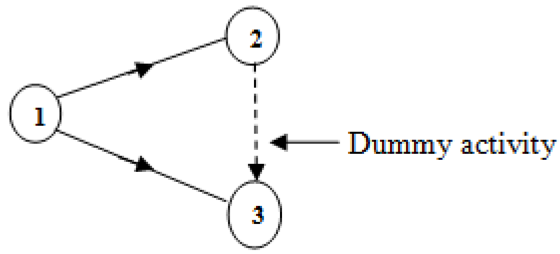
Network analysis is the general name given to certain specific techniques which can be used for the planning, management and control of projects. A fundamental method in both PERT and CPM is the use of network systems as a means of graphically depicting the current problems or proposed projects in network diagram. A network diagram is the first thing to sketch an arrow diagram which shows inter-dependencies and the precedence relationship among activities of the project. Before illustrating the network representation of a project, let us define some basic definitions:

18.4.1 Activity

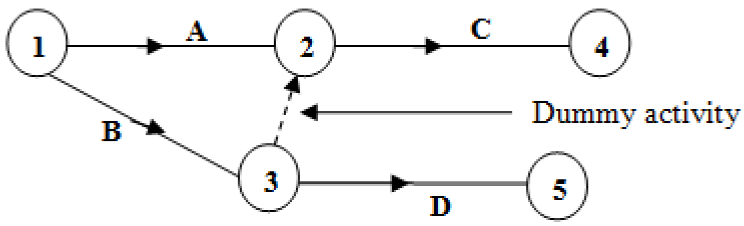
Any individual operation, which utilizes resources and has a beginning and an end is called an activity. An arrow is used to depict an activity with its head indicating the direction of progress in the project. It is of four types:

a) **Predecessor activity:** activity that must be completed immediately prior to the start of another activity.
b) **Successor activity:** activity which cannot be started until one or more of other activities are completed but immediately succeed them are called successor activity.
c) **Concurrent:** Activity which can be accomplished concurrently is known as concurrent activity. An activity can be predecessor or successor to an event or it may be concurrent with the one or more of the other activities.
d) **Dummy activity:** An activity which does not consume any kind of resources but merely depicts the technological dependence is called a dummy activity. Dummy activity is inserted in a network to classify the activity pattern in the following situations:

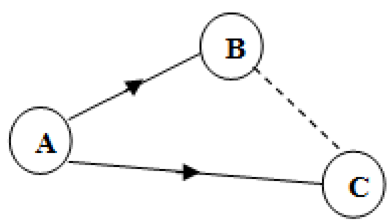
i) To make activities with common starting and finishing points distinguishable.
ii) To identify and maintain the proper precedence relationship between activities those are not connected by events.



Let�s consider a situation where A and B are concurrent activities and activity D is dependent on B and C is dependent on both A and B. Such a situation can be handled by use of dummy activity.



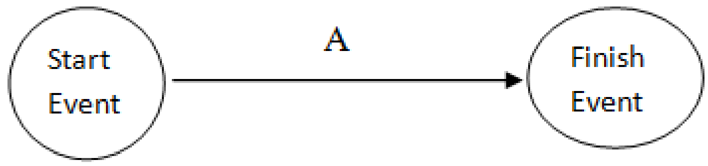
When two or more activities are exactly parallel such that they would start at the same node (event) and finish at the same node. A dummy would be inserted between the end of one of the activities and the common finishing node.



This is to ensure that each activity has a unique description when refer to by its start and finish node number. Dummy are often used to improve the layout of network. When they may not strictly necessary to represents the logic involved. This often happens at the start or finish of a network where a number of activities either start from a certain point or converge to particular point.

18.4.2 Event

The beginning and end points of an activity are called events or nodes or connector. This is usually represented by circle in a network.



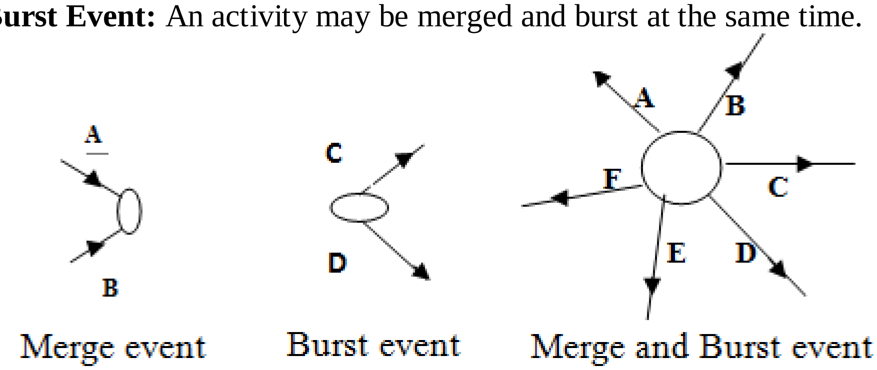
Here, A is known as the activity.

The events can be further classified into three categories:

a) **Merge Event**: When two or more activities come from an event it is known as merge event.

b) **Burst Event:** When more than one activity leaves an event is known as burst event.

c) **Merge & Burst Event:** An activity may be merged and burst at the same time.



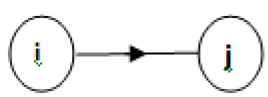
18.4.3 Difference between event and activity

An event is that particular instant of time at which some specific part of project is to be achieved while an activity is the actual performance of a task. An activity requires time and resources for its completion. Events are generally described by such words as complete, start, issue, approves, taste etc. while the word like design, process, test, develop, prepare etc. shows that a work is being accomplished and thus represent activity. While drawing networks, it is assumed that

a) The movement is from left to right and

b) Head event has a number higher than the tail event.

Thus the activity (i-j) always means that job which begins at event (i) is completed at event (j).



Network representation is based on the following two axioms.

a) An event is not said to be complete until all the activities flowing into it are completed.

b) No subsequent activities can begin until its tail event is reached or completed.

2.1 <u>Introduction to CPM / PERT Techniques</u>

CPM/PERT or Network Analysis as the technique is sometimes called, developed along two parallel streams, one industrial and the other military.

CPM (Critical Path Method) was the discovery of M.R.Walker of E.I.Du Pont de Nemours & Co. and J.E.Kelly of Remington Rand, circa 1957. The computation was designed for the UNIVAC-I computer. The first test was made in 1958, when CPM was applied to the construction of a new chemical plant. In March 1959, the method was applied to maintenance shut-down at the Du Pont works in Louisville, Kentucky. Unproductive time was reduced from 125 to 93 hours.

PERT (Project Evaluation and Review Technique) was devised in 1958 for the POLARIS missile program by the Program Evaluation Branch of the Special Projects office of the U.S.Navy, helped by the Lockheed Missile Systems division and the Consultant firm of Booz-Allen & Hamilton. The calculations were so arranged so that they could be carried out on the IBM Naval Ordinance Research Computer (NORC) at Dahlgren, Virginia.

The methods are essentially **network-oriented techniques** using the same principle. PERT and CPM are basically time-oriented methods in the sense that they both lead to determination of a time schedule for the project. The significant difference between two approaches is that the time estimates for the different activities in CPM were assumed to be **deterministic** while in PERT these are described **probabilistically**. These techniques are referred as **project scheduling** techniques.

In **CPM** activities are shown as a network of precedence relationships using activity-on-node network construction

- Single estimate of activity time
- Deterministic activity times

USED IN: Production management - for the jobs of repetitive in nature where the activity time estimates can be predicted with considerable certainty due to the existence of past experience.

In **PERT** activities are shown as a network of precedence relationships using activity-on-arrow network construction

- Multiple time estimates
- Probabilistic activity times

USED IN: Project management - for non-repetitive jobs (research and development work), where the time and cost estimates tend to be quite uncertain. This technique uses probabilistic time estimates.

Benefits of PERT/CPM

- Useful at many stages of project management
- Mathematically simple

- Give critical path and slack time
- Provide project documentation
- Useful in monitoring costs

Limitations of PERT/CPM

- Clearly defined, independent and stable activities
- Specified precedence relationships
- Over emphasis on critical paths

2.2 Applications of CPM / PERT

These methods have been applied to a wide variety of problems in industries and have found acceptance even in government organizations. These include

- Construction of a dam or a canal system in a region
- Construction of a building or highway
- Maintenance or overhaul of airplanes or oil refinery
- Space flight
- Cost control of a project using PERT / COST
- Designing a prototype of a machine
- Development of supersonic planes

2.3 Basic Steps in PERT / CPM

Project scheduling by PERT / CPM consists of four main steps

1. Planning

- The planning phase is started by splitting the total project in to small projects. These smaller projects in turn are divided into activities and are analyzed by the department or section.
- The relationship of each activity with respect to other activities are defined and established and the corresponding responsibilities and the authority are also stated.
- Thus the possibility of overlooking any task necessary for the completion of the project is reduced substantially.

2. Scheduling

- The ultimate objective of the scheduling phase is to prepare a time chart showing the start and finish times for each activity as well as its relationship to other activities of the project.
- Moreover the schedule must pinpoint the critical path activities which require special attention if the project is to be completed in time.
- For non-critical activities, the schedule must show the amount of slack or float times which can be used advantageously when such activities are delayed or when limited resources are to be utilized effectively.

3. Allocation of resources

- Allocation of resources is performed to achieve the desired objective. A resource is a physical variable such as labour, finance, equipment and space which will impose a limitation on time for the project.
- When resources are limited and conflicting, demands are made for the same type of resources a systematic method for allocation of resources become essential.
- Resource allocation usually incurs a compromise and the choice of this compromise depends on the judgment of managers.

4. Controlling

- The final phase in project management is controlling. Critical path methods facilitate the application of the principle of management by expectation to identify areas that are critical to the completion of the project.
- By having progress reports from time to time and updating the network continuously, a better financial as well as technical control over the project is exercised.
- Arrow diagrams and time charts are used for making periodic progress reports. If required, a new course of action is determined for the remaining portion of the project.

2.4 The Framework for PERT and CPM

Essentially, there are six steps which are common to both the techniques. The procedure is listed below:

I. Define the Project and all of its significant activities or tasks. The Project (made up of several tasks) should have only a single start activity and a single finish activity.

II. Develop the relationships among the activities. Decide which activities must precede and which must follow others.

III. Draw the "Network" connecting all the activities. Each Activity should have unique event numbers. Dummy arrows are used where required to avoid giving the same numbering to two activities.

IV. Assign time and/or cost estimates to each activity

V. Compute the longest time path through the network. This is called the critical path.

VI. Use the Network to help plan, schedule, and monitor and control the project.

The Key Concept used by CPM/PERT is that a small set of activities, which make up the longest path through the activity network control the entire project. If these "critical" activities could be identified and assigned to responsible persons, management resources could be optimally used by concentrating on the few activities which determine the fate of the entire project.

Non-critical activities can be replanned, rescheduled and resources for them can be reallocated flexibly, without affecting the whole project.

Five useful questions to ask when preparing an activity network are:

- Is this a Start Activity?
- Is this a Finish Activity?
- What Activity Precedes this?
- What Activity Follows this?
- What Activity is Concurrent with this?

2.5 Network Diagram Representation

In a network representation of a project certain definitions are used

1. Activity

Any individual operation which utilizes resources and has an end and a beginning is called activity. An arrow is commonly used to represent an activity with its head indicating the direction of progress in the project. These are classified into four categories

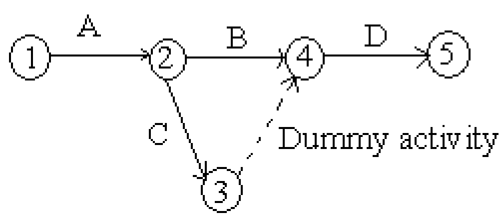
1. **Predecessor activity** – Activities that must be completed immediately prior to the start of another activity are called predecessor activities.

2. **Successor activity** – Activities that cannot be started until one or more of other activities are completed but immediately succeed them are called successor activities.
3. **Concurrent activity** – Activities which can be accomplished concurrently are known as concurrent activities. It may be noted that an activity can be a predecessor or a successor to an event or it may be concurrent with one or more of other activities.
4. **Dummy activity** – An activity which does not consume any kind of resource but merely depicts the technological dependence is called a dummy activity.

The dummy activity is inserted in the network to clarify the activity pattern in the following two situations

- To make activities with common starting and finishing points distinguishable
- To identify and maintain the proper precedence relationship between activities that is not connected by events.

For example, consider a situation where A and B are concurrent activities. C is dependent on A and D is dependent on A and B both. Such a situation can be handled by using a dummy activity as shown in the figure.



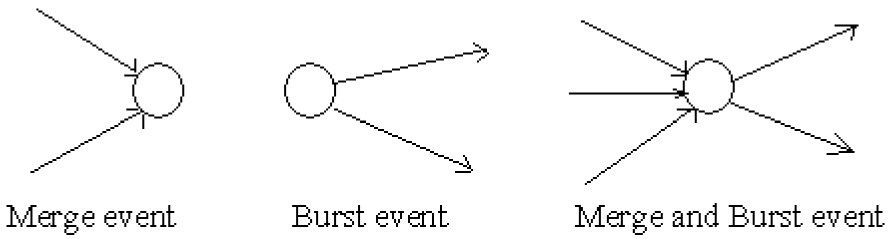
2. Event

An event represents a point in time signifying the completion of some activities and the beginning of new ones. This is usually represented by a circle in a network which is also called a node or connector.

The events are classified in to three categories

1. **Merge event** – When more than one activity comes and joins an event such an event is known as merge event.

2. **Burst event** – When more than one activity leaves an event such an event is known as burst event.
3. **Merge and Burst event** – An activity may be merge and burst event at the same time as with respect to some activities it can be a merge event and with respect to some other activities it may be a burst event.



3. Sequencing

The first prerequisite in the development of network is to maintain the precedence relationships. In order to make a network, the following points should be taken into considerations

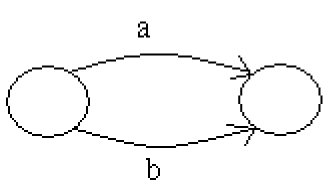
- What job or jobs precede it?
- What job or jobs could run concurrently?
- What job or jobs follow it?
- What controls the start and finish of a job?

Since all further calculations are based on the network, it is necessary that a network be drawn with full care.

2.6 Rules for Drawing Network Diagram

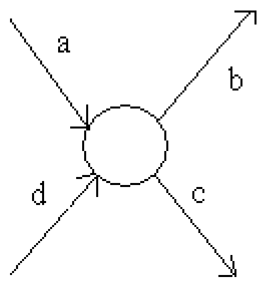
Rule 1

Each activity is represented by one and only one arrow in the network



Rule 2

No two activities can be identified by the same end events



Rule 3

In order to ensure the correct precedence relationship in the arrow diagram, following questions must be checked whenever any activity is added to the network

- What activity must be completed immediately before this activity can start?
- What activities must follow this activity?
- What activities must occur simultaneously with this activity?

In case of large network, it is essential that certain good habits be practiced to draw an easy to follow network

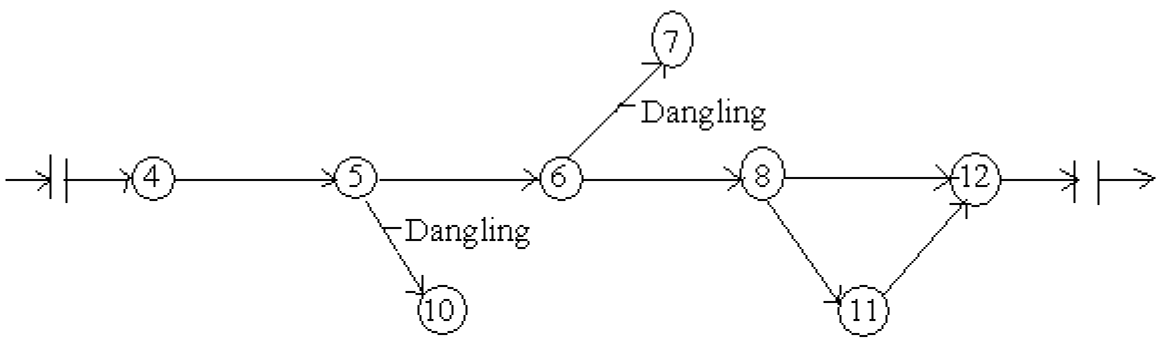
- Try to avoid arrows which cross each other
- Use straight arrows
- Do not attempt to represent duration of activity by its arrow length
- Use arrows from left to right. Avoid mixing two directions, vertical and standing arrows may be used if necessary.
- Use dummies freely in rough draft but final network should not have any redundant dummies.
- The network has only one entry point called start event and one point of emergence called the end event.

2.7 <u>Common Errors in Drawing Networks</u>

The three types of errors are most commonly observed in drawing network diagrams

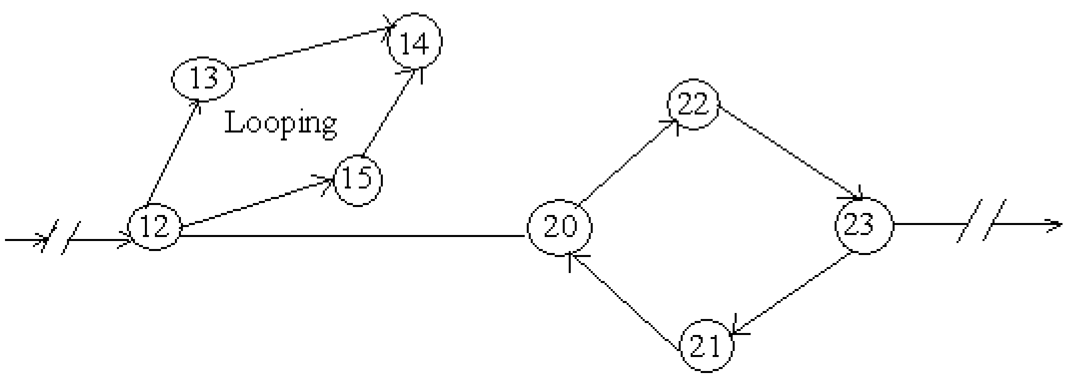
1. Dangling

To disconnect an activity before the completion of all activities in a network diagram is known as dangling. As shown in the figure activities (5 – 10) and (6 – 7) are not the last activities in the network. So the diagram is wrong and indicates the error of dangling



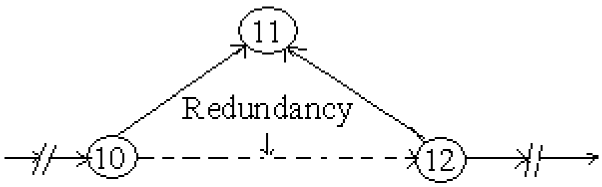
2. Looping or Cycling

Looping error is also known as cycling error in a network diagram. Drawing an endless loop in a network is known as error of looping as shown in the following figure.



3. Redundancy

Unnecessarily inserting the dummy activity in network logic is known as the error of redundancy as shown in the following diagram



2.8 <u>Advantages and Disadvantages</u>

PERT/CPM has the following advantages

- A PERT/CPM chart explicitly defines and makes visible dependencies (precedence relationships) between the elements,
- PERT/CPM facilitates identification of the critical path and makes this visible,
- PERT/CPM facilitates identification of early start, late start, and slack for each activity,
- PERT/CPM provides for potentially reduced project duration due to better understanding of dependencies leading to improved overlapping of activities and tasks where feasible.

PERT/CPM has the following disadvantages:

- There can be potentially hundreds or thousands of activities and individual dependency relationships,
- The network charts tend to be large and unwieldy requiring several pages to print and requiring special size paper,
- The lack of a timeframe on most PERT/CPM charts makes it harder to show status although colours can help (e.g., specific colour for completed nodes),
- When the PERT/CPM charts become unwieldy, they are no longer used to manage the project.

2.9 Critical Path in Network Analysis

Basic Scheduling Computations

The notations used are

(i, j) = Activity with tail event i and head event j

E_i = Earliest occurrence time of event i

L_j = Latest allowable occurrence time of event j

D_{ij} = Estimated completion time of activity (i, j)

$(Es)_{ij}$ = Earliest starting time of activity (i, j)

$(Ef)_{ij}$ = Earliest finishing time of activity (i, j)

$(Ls)_{ij}$ = Latest starting time of activity (i, j)

$(Lf)_{ij}$ = Latest finishing time of activity (i, j)

The procedure is as follows

1. Determination of Earliest time (E_j): Forward Pass computation

- **Step 1**

  The computation begins from the start node and move towards the end node. For easiness, the forward pass computation starts by assuming the earliest occurrence time of zero for the initial project event.

- Step 2

  i. Earliest starting time of activity (i, j) is the earliest event time of the tail end event i.e. $(Es)_{ij} = E_i$

  ii. Earliest finish time of activity (i, j) is the earliest starting time + the activity time i.e. $(Ef)_{ij} = (Es)_{ij} + D_{ij}$ or $(Ef)_{ij} = E_i + D_{ij}$

iii. Earliest event time for event j is the maximum of the earliest finish times of all activities ending in to that event i.e. E_j = max [$(Ef)_{ij}$ for all immediate predecessor of (i, j)] or E_j =max [$E_i + D_{ij}$]

2. Backward Pass computation (for latest allowable time)

- **Step 1**

  For ending event assume E = L. Remember that all E's have been computed by forward pass computations.

- Step 2

  Latest finish time for activity (i, j) is equal to the latest event time of event j i.e. $(Lf)_{ij} = L_j$

- Step 3

  Latest starting time of activity (i, j) = the latest completion time of (i, j) – the activity time or $(Ls)_{ij} = (Lf)_{ij} - D_{ij}$ or $(Ls)_{ij} = L_j - D_{ij}$

- Step 4

  Latest event time for event 'i' is the minimum of the latest start time of all activities originating from that event i.e. L_i = min [$(Ls)_{ij}$ for all immediate successor of (i, j)] = min [$(Lf)_{ij} - D_{ij}$] = min [$L_j - D_{ij}$]

3. Determination of floats and slack times

There are three kinds of floats

- **Total float** – The amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time without affecting the overall project duration time.

  Mathematically

  $(Tf)_{ij}$ = (Latest start – Earliest start) for activity (i – j)

  $(Tf)_{ij} = (Ls)_{ij} - (Es)_{ij}$ or $(Tf)_{ij} = (L_j - D_{ij}) - E_i$

- **Free float –** The time by which the completion of an activity can be delayed beyond the earliest finish time without affecting the earliest start of a subsequent activity.

  Mathematically

  $(Ff)_{ij}$ = (Earliest time for event j – Earliest time for event i) – Activity time for (i, j)

  $(Ff)_{ij} = (E_j - E_i) - D_{ij}$

- **Independent float** – The amount of time by which the start of an activity can be delayed without effecting the earliest start time of any immediately following activities, assuming that the preceding activity has finished at its latest finish time.

  Mathematically

  $(If)_{ij} = (E_j - L_i) - D_{ij}$

  The negative independent float is always taken as zero.

- **Event slack** - It is defined as the difference between the latest event and earliest event times.

  Mathematically

  Head event slack = $L_j – E_j$, Tail event slack = $L_i - E_i$

4. Determination of critical path

- **Critical event** – The events with zero slack times are called critical events. In other words the event i is said to be critical if $E_i = L_i$

- **Critical activity** – The activities with zero total float are known as critical activities. In other words an activity is said to be critical if a delay in its start will cause a further delay in the completion date of the entire project.

- **Critical path** – The sequence of critical activities in a network is called critical path. The critical path is the longest path in the network from the starting event to ending event and defines the minimum time required to complete the project.

Exercise

1. What is PERT and CPM?
2. What are the advantages of using PERT/CPM?
3. Mention the applications of PERT/CPM
4. Explain the following terms
    a. Earliest time
    b. Latest time
    c. Total activity slack
    d. Event slack
    e. Critical path
5. Explain the CPM in network analysis.
6. What are the rules for drawing network diagram? Also mention the common errors that occur in drawing networks.
7. What is the difference between PERT and CPM/

8. What are the uses of PERT and CPM?
9. Explain the basic steps in PERT/CPM techniques.
10. Write the framework of PERT/CPM.

Unit 3

3.1 Worked Examples on CPM

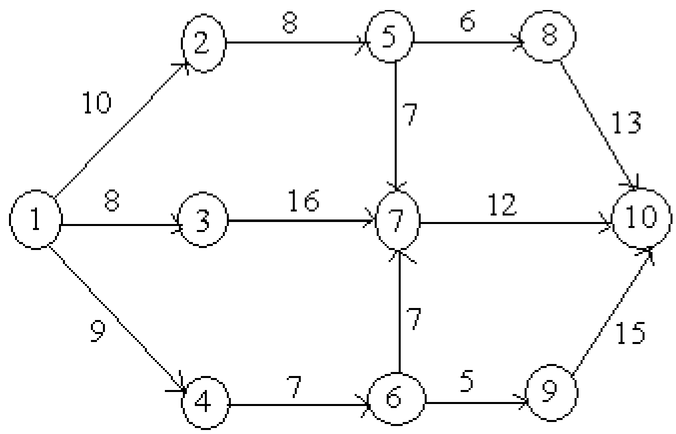
3.2 PERT

3.3 Worked Examples

3.1 Worked Examples on CPM

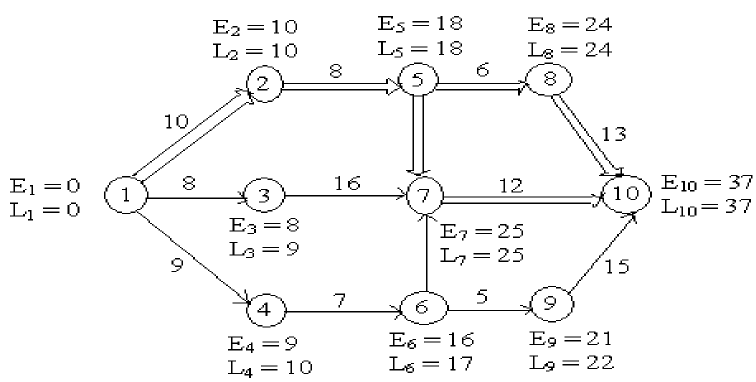
Example 1

Determine the early start and late start in respect of all node points and identify critical path for the following network.



Solution

Calculation of E and L for each node is shown in the network



Activity(i, j)	Normal Time (D_{ij})	Earliest Time		Latest Time		Float Time (L_i - D_{ij}) - E_i
		Start (E_i)	Finish (E_i + D_{ij})	Start (L_i - D_{ij})	Finish (L_i)	
(1, 2)	10	0	10	0	10	0
(1, 3)	8	0	8	1	9	1
(1, 4)	9	0	9	1	10	1
(2, 5)	8	10	18	10	18	0
(4, 6)	7	9	16	10	17	1

(3, 7)	16	8	24	9	25	1
(5, 7)	7	18	25	18	25	0
(6, 7)	7	16	23	18	25	2
(5, 8)	6	18	24	18	24	0
(6, 9)	5	16	21	17	22	1
(7, 10)	12	25	37	25	37	0
(8, 10)	13	24	37	24	37	0
(9, 10)	15	21	36	22	37	1

Network Analysis Table

From the table, the critical nodes are (1, 2), (2, 5), (5, 7), (5, 8), (7, 10) and (8, 10)

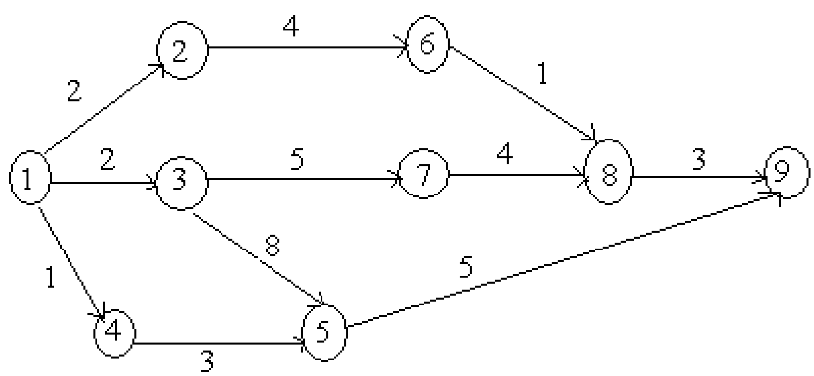
From the table, there are two possible critical paths

i. $1 \rightarrow 2 \rightarrow 5 \rightarrow 8 \rightarrow 10$

ii. $1 \rightarrow 2 \rightarrow 5 \rightarrow 7 \rightarrow 10$

Example 2

Find the critical path and calculate the slack time for the following network

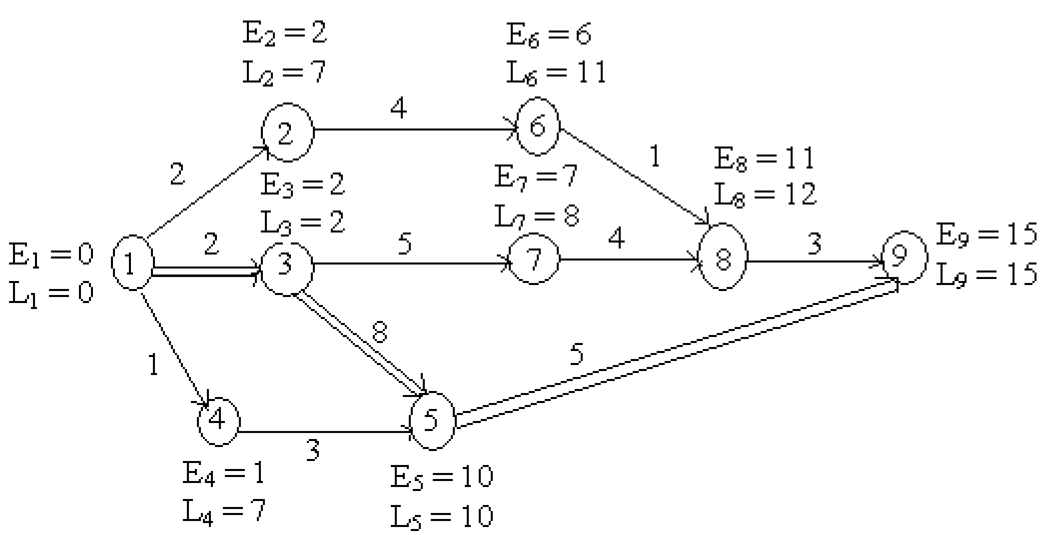


Solution

The earliest time and the latest time are obtained below

Activity(i, j)	Normal Time (D_{ij})	Earliest Time		Latest Time		Float Time (L_i - D_{ij}) - E_i
		Start (E_i)	Finish (E_i + D_{ij})	Start (L_i - D_{ij})	Finish (L_i)	
(1, 2)	2	0	2	5	7	5
(1, 3)	2	0	2	0	2	0
(1, 4)	1	0	1	6	7	6
(2, 6)	4	2	6	7	11	5
(3, 7)	5	2	7	3	8	1
(3, 5)	8	2	10	2	10	0
(4, 5)	3	1	4	7	10	6
(5, 9)	5	10	15	10	15	0
(6, 8)	1	6	7	11	12	5
(7, 8)	4	7	11	8	12	1
(8, 9)	3	11	14	12	15	1

From the above table, the critical nodes are the activities (1, 3), (3, 5) and (5, 9)



The critical path is 1 → 3 → 5 → 9

Example 3

A project has the following times schedule

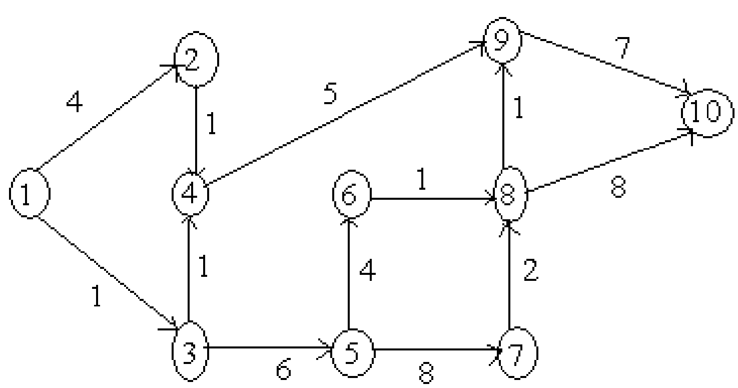
Activity	Times in weeks	Activity	Times in weeks
(1 – 2)	4	(5 – 7)	8
(1 – 3)	1	(6 – 8)	1
(2 – 4)	1	(7 – 8)	2
(3 – 4)	1	(8 – 9)	1
(3 – 5)	6	(8 – 10)	8
(4 – 9)	5	(9 – 10)	7
(5 – 6)	4		

Construct the network and compute

1. T_E and T_L for each event
2. Float for each activity
3. Critical path and its duration

Solution

The network is

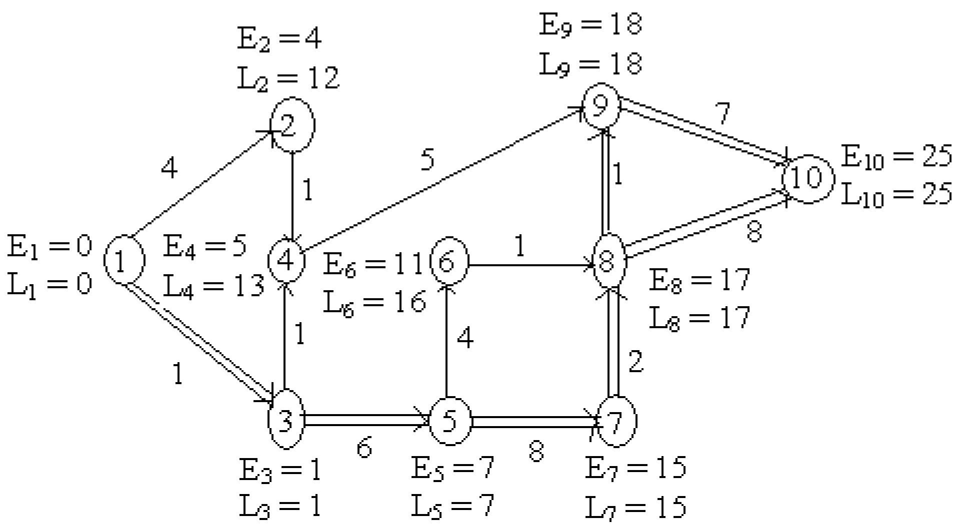


Event No.:	1	2	3	4	5	6	7	8	9	10
T_E:	0	4	1	5	7	11	15	17	18	25
T_L:	0	12	1	13	7	16	15	17	18	25

Float = T_L (Head event) – T_E (Tail event) – Duration

Activity	Duration	T_E (Tail event)	T_L (Head event)	Float
(1 – 2)	4	0	12	8
(1 – 3)	1	0	1	0
(2 – 4)	1	4	13	8
(3 – 4)	1	1	13	11
(3 – 5)	6	1	7	0
(4 – 9)	5	5	18	8
(5 – 6)	4	7	16	5
(5 – 7)	8	7	15	0
(6 – 8)	1	11	17	5
(7 – 8)	2	15	17	0
(8 – 9)	1	17	18	0
(8 – 10)	8	17	25	0
(9 – 10)	7	18	25	0

The resultant network shows the critical path



The two critical paths are

i. $1 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10$

ii. $1 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 8 \rightarrow 10$

3.2 Project Evaluation and Review Technique (PERT)

The main objective in the analysis through PERT is to find out the completion for a particular event within specified date. The PERT approach takes into account the uncertainties. The three time values are associated with each activity

1. **Optimistic time** – It is the shortest possible time in which the activity can be finished. It assumes that every thing goes very well. This is denoted by t_0.
2. **Most likely time** – It is the estimate of the normal time the activity would take. This assumes normal delays. If a graph is plotted in the time of completion and the frequency of completion in that time period, then most likely time will represent the highest frequency of occurrence. This is denoted by t_m.
3. **Pessimistic time** – It represents the longest time the activity could take if everything goes wrong. As in optimistic estimate, this value may be such that

only one in hundred or one in twenty will take time longer than this value. This is denoted by t_p.

In PERT calculation, all values are used to obtain the percent expected value.

1. **Expected time** – It is the average time an activity will take if it were to be repeated on large number of times and is based on the assumption that the activity time follows Beta distribution, this is given by

$$t_e = (t_0 + 4\ t_m + t_p) / 6$$

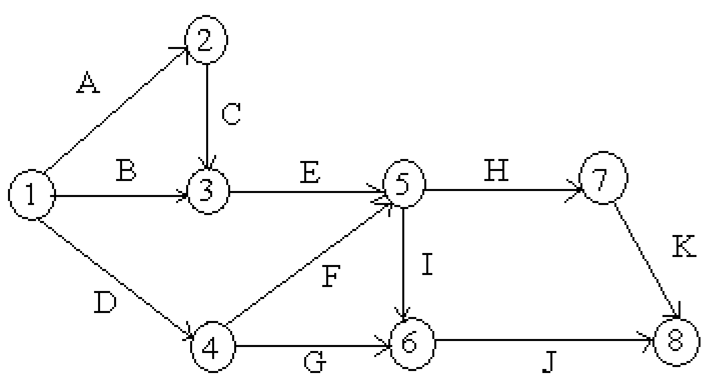
2. The **variance** for the activity is given by

$$\sigma^2 = [(t_p - t_o) / 6]^2$$

3.3 Worked Examples

Example 1

For the project



Task:	A	B	C	D	E	F	G	H	I	J	K
Least time:	4	5	8	2	4	6	8	5	3	5	6

Greatest time: 8 10 12 7 10 15 16 9 7 11 13

Most likely time: 5 7 11 3 7 9 12 6 5 8 9

Find the earliest and latest expected time to each event and also critical path in the network.

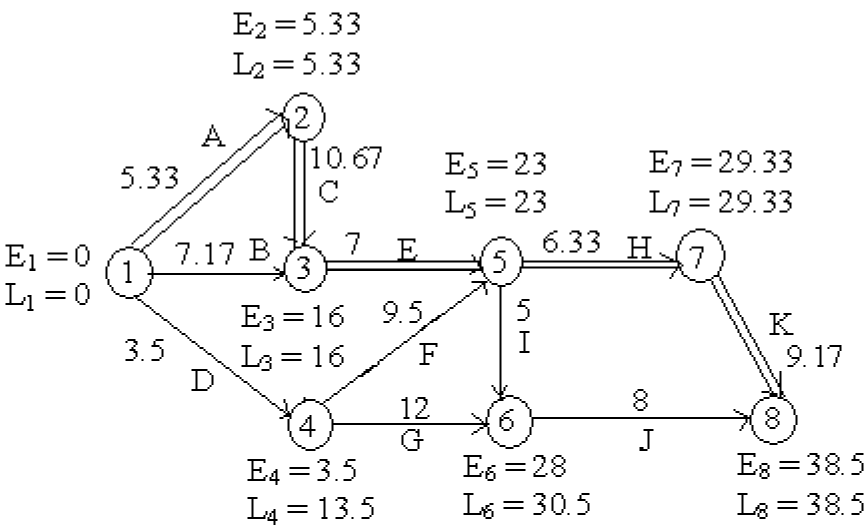
Solution

Task	Least time(t_0)	Greatest time (t_p)	Most likely time (t_m)	Expected time ($t_o + t_p + 4t_m$)/6
A	4	8	5	5.33
B	5	10	7	7.17
C	8	12	11	10.67
D	2	7	3	3.5
E	4	10	7	7
F	6	15	9	9.5
G	8	16	12	12
H	5	9	6	6.33
I	3	7	5	5
J	5	11	8	8
K	6	13	9	9.17

Task	Expected time (t_e)	Start		Finish		Total float
		Earliest	Latest	Earliest	Latest	
A	5.33	0	0	5.33	5.33	0
B	7.17	0	8.83	7.17	16	8.83
C	10.67	5.33	5.33	16	16	0
D	3.5	0	10	3.5	13.5	10
E	7	16	16	23	23	0

F	9.5	3.5	13.5	13	23	10
G	12	3.5	18.5	15.5	30.5	15
H	6.33	23	23	29.33	29.33	0
I	5	23	25.5	28	30.5	2.5
J	8	28	30.5	36	38.5	2.5
K	9.17	29.33	29.33	31.5	38.5	0

The network is



The critical path is A →C →E → H → K

Example 2

A project has the following characteristics

Activity	Most optimistic time (a)	Most pessimistic time (b)	Most likely time (m)
(1 – 2)	1	5	1.5
(2 – 3)	1	3	2
(2 – 4)	1	5	3

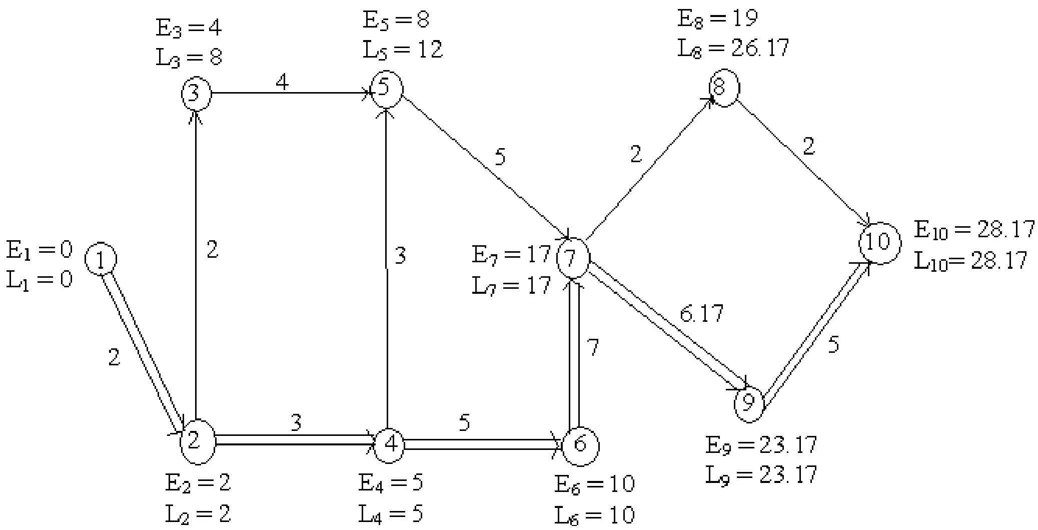
(3 – 5)	3	5	4
(4 – 5)	2	4	3
(4 – 6)	3	7	5
(5 – 7)	4	6	5
(6 – 7)	6	8	7
(7 – 8)	2	6	4
(7 – 9)	5	8	6
(8 – 10)	1	3	2
(9 – 10)	3	7	5

Construct a PERT network. Find the critical path and variance for each event.

Solution

Activity	(a)	(b)	(m)	(4m)	t_e (a + b + 4m)/6	v $[(b – a) / 6]^2$
(1 – 2)	1	5	1.5	6	2	4/9
(2 – 3)	1	3	2	8	2	1/9
(2 – 4)	1	5	3	12	3	4/9
(3 – 5)	3	5	4	16	4	1/9
(4 – 5)	2	4	3	12	3	1/9
(4 – 6)	3	7	5	20	5	4/9
(5 – 7)	4	6	5	20	5	1/9
(6 – 7)	6	8	7	28	7	1/9
(7 – 8)	2	6	4	16	4	4/9
(7 – 9)	5	8	6	24	6.17	1/4
(8 – 10)	1	3	2	8	2	1/9
(9 – 10)	3	7	5	20	5	4/9

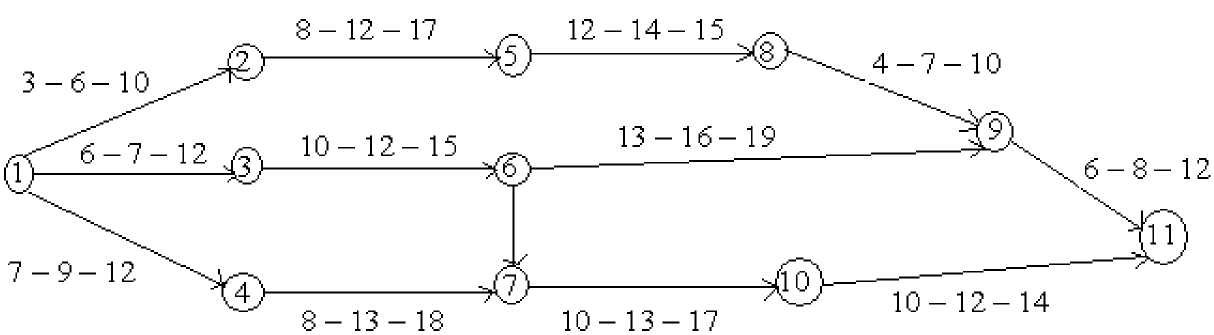
The network is constructed as shown below



The critical path = 1 → 2 → 4 → 6 → 7 →9 →10

Example 3

Calculate the variance and the expected time for each activity



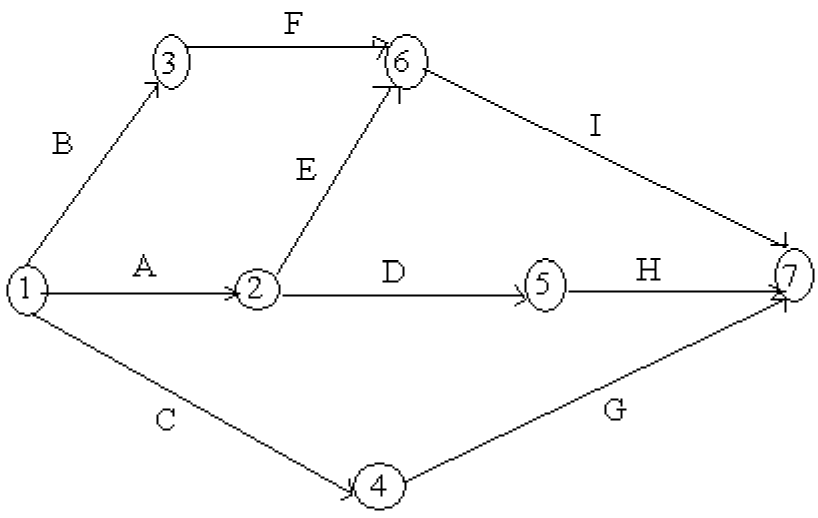
Solution

Activity	(t_o)	(t_m)	(t_p)	t_e $(t_o + t_p + 4t_m)/6$	v $[(t_p – t_o) / 6]^2$
(1 – 2)	3	6	10	6.2	1.36
(1 – 3)	6	7	12	7.7	1.00
(1 – 4)	7	9	12	9.2	0.69

(2 – 3)	0	0	0	0.0	0.00
(2 – 5)	8	12	17	12.2	2.25
(3 – 6)	10	12	15	12.2	0.69
(4 – 7)	8	13	19	13.2	3.36
(5 – 8)	12	14	15	13.9	0.25
(6 – 7)	8	9	10	9.0	0.11
(6 – 9)	13	16	19	16.0	1.00
(8 – 9)	4	7	10	7.0	1.00
(7 – 10)	10	13	17	13.2	1.36
(9 – 11)	6	8	12	8.4	1.00
(10 – 11)	10	12	14	12.0	0.66

Example 4

A project is represented by the network as shown below and has the following data



Task:	A	B	C	D	E	F	G	H	I
Least time:	5	18	26	16	15	6	7	7	3

Greatest time:	10	22	40	20	25	12	12	9	5
Most likely time:	15	20	33	18	20	9	10	8	4

Determine the following

1. Expected task time and their variance
2. Earliest and latest time

Solution

1.

Activity	Least time (t_0)	Greatest time (t_p)	Most likely time (t_m)	Expected time ($t_o + t_p + 4t_m$)/6	Variance (σ^2)
(1-2)	5	10	8	7.8	0.69
(1-3)	18	22	20	20.0	0.44
(1-4)	26	40	33	33.0	5.43
(2-5)	16	20	18	18.0	0.44
(2-6)	15	25	20	20.0	2.78
(3-6)	6	12	9	9.0	1.00
(4-7)	7	12	10	9.8	0.69
(5-7)	7	9	8	8.0	0.11
(6-7)	3	5	4	4.0	0.11

2.

Earliest time

$E_1 = 0$

$E_2 = 0 + 7.8 = 7.8$

$E_3 = 0 + 20 = 20$

$E_4 = 0 + 33 = 33$

$E_5 = 7.8 + 18 = 25.8$

$E_6 = \max [7.8 + 20, 20 + 9] = 29$

$E_7 = \max [33 + 9.8, 25.8 + 8, 29 + 4] = 42.8$

Latest time

$L_7 = 42.8$

$L_6 = 42.8 – 4 = 38.8$

$L_5 = 42.8 – 8 = 34.3$

$L_4 = 42.8 – 9.8 = 33$

$L_3 = 38.8 – 9 = 29.8$

$L_2 = \min [34.8 – 18, 38.8 – 20] = 16.8$

$L_1 = \min [16.8 – 7.8, 29.8 – 20, 33 - 33] = 0$

Exercise

1. What is PERT?

2. For the following data, draw network. Find the critical path, slack time after calculating the earliest expected time and the latest allowable time

Activity	Duration	Activity	Duration
(1 – 2)	5	(5 – 9)	3
(1 – 3)	8	(6 – 10)	5
(2 – 4)	6	(7 – 10)	4
(2 – 5)	4	(8 – 11)	9
(2 – 6)	4	(9 – 12)	2
(3– 7)	5	(10 – 12)	4
(3 – 8)	3	(11 – 13)	1
(4 – 9)	1	(12 – 13)	7

[Ans. Critical path: 1 → 3 → 7 → 10 → 12 →13]

3. A project schedule has the following characteristics

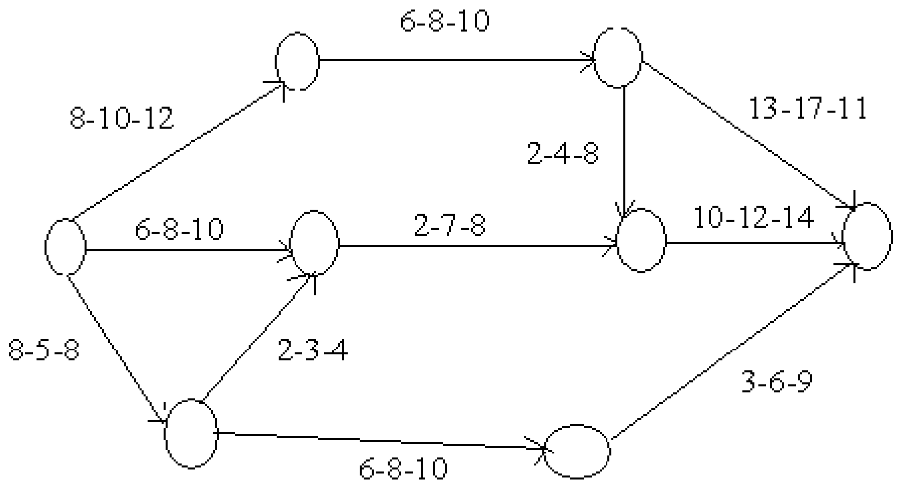
Activity	Most optimistic time	Most likely time	Most pessimistic time
(1 – 2)	1	2	3
(2 – 3)	1	2	3
(2 – 4)	1	3	5
(3 – 5)	3	4	5
(4 – 5)	2	5	4
(4 – 6)	3	5	7
(5 – 7)	4	5	6
(6 – 7)	6	7	8
(7 – 8)	2	4	6

(7 – 9)	4	6	8
(8 – 10)	1	2	3
(9 – 10)	3	5	7

Construct a PERT network and find out

a. The earliest possible time
b. Latest allowable time
c. Slack values
d. Critical path

4. Explain the following terms
   a. optimistic time
   b. Most likely time
   c. Pessimistic time
   d. Expected time
   e. Variance
5. Calculate the variance and the expected time for each activity



UNIT-IV

Replacement Model:

Replacement of Items that deteriorate whose maintenance costs increase with time without change in the money value.

Replacement of items that fail suddenly:
individual replacement policy,
group replacement policy.

Inventory Models:

Models with deterministic demand model
(a) demand rate uniform and production rate infinite, model
(b) demand rate non-uniform and production rate infinite, model
(c) demand rate uniform and production rate finite.

REPLACEMENT MODELS

INTRODUCTION

The problem of replacement is felt when the job performing units such as men, machines, equipments, parts etc. become less effective or useless due to either sudden or gradual deterioration in their efficiency, failure or breakdown. By replacing them with new ones at frequent intervals, maintenance and other overhead costs can be reduced. However, such replacements would increase the need of capital cost for new ones.

For example,

1. A vehicle tends to wear out with time due to constant use. More money needs to be spent on it on account of increased repair and operating cost. A stage comes when it becomes uneconomical to maintain the vehicle and it is better to replace it with a new one. Here the replacement decision may be taken to balance the increasing maintenance cost with the decreasing money value of the vehicle as time passes.
2. In case of highway tubelights where time of failure is not predictable for individual tubes, replacement is done only after individual failure. However, it may be economical to replace such tubes on a schedule basis before their failure. Here the replacement decision may be taken to balance between the wasted life of a tube before failure and cost incurred when a tube fails completely during service.

Thus, the basic problem in such situations is to formulate a replacement policy to determine an age (or period) at which the replacement of the given machinery/equipment is most economical keeping in view all possible alternatives.

In this chapter, we shall discuss the replacement policies in the context of the following three types of replacement situations:

1. Items such as machines, vehicles, tyres etc. whose efficiency deteriorates with age due to constant use and which need increased operating and maintenance costs. In such cases deterioration level is predictable and is represented by a. Increased maintenance/operational cost, b. its waste or scrap value and damage to item and safety risk.
2. Items such as light bulbs and tubes, electric motors, radio, television parts etc. which do not give any indication of deteriorations with time but fail all of a sudden and become completely useless. Such cases require an anticipation of failures to specify the probability of failure for any future time period. With this probability distribution and the cost information, it is desired to formulate optimal, replacement policy to balance the wasted life of an item replaced before failure against the costs incurred when the item fail in service.
3. The existing working staff in an organisation gradually reduces due to retirement, death, retrenchment and other reasons.

TYPES OF FAILURE

The term 'failure' here will be discussed in the context of replacement decisions. There are two types of failures: 1. Gradual failure and 2. Sudden failure.

Gradual Failure

Gradual Failure is progressive in nature. That is, as the life of an item increases its operational efficiency also deteriorates resulting in

- increased running (maintenance and operating) costs.
- decrease in its productivity.

- decrease in the resale or salvage value.

Mechanical items like pistons, rings, bearings etc., and automobile types fall under this category.

Sudden Failure

This type of failure occurs in items after some period of giving desired service rather than deterioration while in service. The period of desired service is not constant but follows some frequency distribution which may be progressive, retrogressive or random in nature.

1. Progressive Failure: If the probability of failure of an item increases with the increase in its life, then such failure is called progressive failure. For example, light bulbs and tubes fail progressively.

2. Retrogressive Failure: If the probability of failure in the beginning of the life of an item is more but as time passes the chances of its failure becomes less, then such failure is said to be retrogressive.

3. Random Failure: In this type of failure, the constant probability of failure is associated with items that fail from random cases such as physical shocks, not related to age. For example, vacuum tubes in air-born equipment have been found to fail at a rate independent of the age of the tube.

REPLACEMENT OF ITEMS WHOSE EFFICIENCY DETERIORATES WITH TIME

When operational efficiency of an item deteriorates with time (gradual failure), it is economical to replace the same with a new one. For example, the maintenance cost of a machine increases with time and a stage is reached when it may not be economical to allow machine to continue in the system. Besides, there could be a number of alternative choices and one may like to compare available alternatives on the basis of the running costs (average maintenance and operating costs) involved. In this section, we shall discuss various techniques for making such comparisons under different conditions. While making such comparisons it is assumed that suitable expressions for running costs are available.

Model 1 Replacement Policy for items Whose Running Cost Increases with Time and Value of Money Remains Constant During a Period

Theorem 1

The cost of maintenance of a machine is given as a function increasing with time and its scrap value is constant.

a. If time is measured continuously, then the average annual cost will be minimised by replacing the machine when the average cost to date becomes equal to the current maintenance cost.

b. If time is measured in discrete units, then the average annual cost will be minimised by replacing the machine when the next period's maintenance cost becomes greater than the current average cost.

PROOF: The aim here is to determine the optimal replacement age of an equipment whose running cost increases with time and the value of money remains constant (i.e. value is not counted) during that period.

Let C = capital or purchase cost of new equipment

S = Scrap (or salvage) value of the equipment at the end of t years

R(t) = running cost of equipment for the year t

n = replacement age of the equipment.

i. When time 't' is a continuous variable: If the equipment is used for t years, then the total cost incurred over this period is given by

TC = Capital (or purchase) cost - Scrap value at the end of t years + Running cost for t years.

$= C - S + \int_0^n R(t)dt$

Therefore, the average cost per unit time incurred over the period of n years is:

$$ATC_n = \frac{1}{n}\left\{C - S + \int_0^n R(t)dt\right\} \text{ ------------------------ (1)}$$

To obtain optimal value n for which ATC_n is minimum, differentiate ATC_n with respect to n, and set the first derivative equal to zero. That is, for minimum of ATC_n.

$$\frac{d}{dn}\{ATC_n\} = -\frac{1}{n2}\{C - S\} + \frac{R(n)}{n} - \frac{1}{n2}\int_0^n R(t)dt = 0$$

$$\text{or } R(n) = \frac{1}{n}\left\{C - S + \int_0^n R(t)dt\right\}, \; n \neq 0 \text{ ------------------ (2)}$$

$R(n) = ATC_n$

Hence, the following replacement policy can be derived with help of Eq. 2.

Policy

Replace the equipment when the average annual cost for n years becomes equal to the current /annual running cost. That is,

$$R(n) = \frac{1}{n}\left\{C - S + \int_0^n R(t)dt\right\}$$

b. When time 't' is a discrete variable: The average cost incurred over the period n is given by

$$ATC_n = \frac{1}{n}\left\{C - S + \sum_{t=0}^{n} R(t)\right\} \text{ ---------------------------(3)}$$

If $C - S$ and $\sum_{t=0}^{n} R(t)$ are assumed to be monotonically decreasing and increasing respectively, then there will exist a value of n for which ATC_n is minimum. Thus we shall have inequalities

$ATC_{n-1} > ATC_n < ATC_{n+1}$

or $ATC_{n-1} - ATC_n > 0$

and $ATC_{n+1} - ATC_n > 0$

Eq. (3) for period n + 1, we get

$$ATC_n = \frac{1}{n+1}\left\{C - S + \sum_{t=1}^{n=0} R(t)\right\} = \frac{1}{n+1}\left\{C - S + \sum_{t=1}^{n=0} R(t) + R(n+1)\right\}$$

$$= \frac{n\ C - S + \sum_{t=0}^{n} R(t)}{n+1 \quad n} + \frac{R(n+1)}{n+1} = \frac{n}{n+1} \cdot ATC_n + \frac{R(n+1)}{n+1}$$

Therefore,

$$ATC_{n+1} - ATC_n = \frac{n}{n+1} ATC_n + \frac{A(n+1)}{n+1} - ATC_n = \frac{R(n+1)}{n+1} + ATC_n\left[\frac{}{n+1} - 1\right] = \frac{R(n+1)}{n+1} - \frac{ATCn}{n+1}$$

Since $ATC_{n+1} - ATC_n > 0$, we get

$$\frac{R(n+1)}{n+1} - \frac{ATCn}{n+1} > 0$$

$R(n+1) - ATC_n > 0$ or $R(n+1) > ATC_n$ -- (4)

Similarly,

$ATC_{n-1} - ATC_n > 0$, implies that $R(n+1) < ATC_{n-1}$. This provides the following replacement policy.

Policy 1: If the next year, running cost R(n + 1) is more than average cost of nth year, ATC_n, then it is economical to replace at the end of n years. That is

$$R(n+1) > \frac{1}{n}\left\{C - S + \sum_{t=0}^{n} R(t)\right\}$$

Policy 2: If the present year's running cost is less than the previous year's average cost, ATC_{n-1}, then do not replace. That is

$$R(n) < \frac{1}{n-1}\left\{C - S + \sum_{t=0}^{n-1} R(t)\right\}$$

EXAMPLE

A firm is considering replacement of a machine, whose cost price is Rs. 12, 200 and scrap value is Rs. 200. The running (maintenance and operating) costs are found from experience to be as follows:

Year	1	2	3	4	5	6	7	8
Running cost (Rs.)	200	500	800	1200	1800	2500	3200	4000

When should the machine be replaced?

Solution: We are given the running cost, R(n), the scrape value S - Rs. 200 and the cost of the machine, C = Rs. 12, 200. In order to determine the optimal time n when the machine should be replaced, first we calculate average cost per year during the life of the machine as shown in table below:

TABLE

Year of Service n	Running Cost (Rs.) R(n)	Cumulative Running Cost (Rs.) Σ R(n)	Depreciation Cost (Rs.) C - S	Total Cost (Rs.) TC	Average Cost (Rs.) ATC_n
1	2	3	4	5 = 3 + 4	6 = 5/1
1	200	200	12000	12, 200	12, 200
2	500	700	12000	12, 700	6350
3	800	1500	12000	13, 500	4500
4	1200	2700	12000	14, 700	3675
5	1800	4500	12000	16, 500	3300
6	2500	7000	12000	19, 000	3167
7	3200	10200	12000	22, 200	3171
8	4000	14200	12000	26, 200	3275

From the table, it may be noted that the average cost per year, ATC_n is minimum in the sixth year (Rs. 3, 167). Also the average cost in the seventh year (Rs. 3171) is more than the cost in the sixth year. Hence, the machine should be replaced after every six years.

(2) The data collected in running a machine, the cost of which is Rs. 60, 000 are given below:

Year	1	2	3	4	5
Resale Value (Rs.)	42, 000	30, 000	20, 400	14, 400	9, 650
Cost of Spares (Rs.)	4, 000	4, 270	4, 880	5, 700	6, 800
Cost of Labour (Rs.)	14, 000	16, 000	18, 000	21, 000	25, 000

Determine the optimum period for replacement of the machine.

Solution: The costs of spares and labour together determine running (operational or maintenance) cost. Thus, the running costs and the resale price of the machine in successive years are as follows:

Year:	1	2	3	4	5
Resale Value (Rs.)	42, 000	30, 000	20, 400	14, 400	9, 650
Running Cost (Rs.)	18, 000	20, 270	22, 880	26, 700	31, 800

The calculations of average running cost per year during the life of the machine are shown in table 1.

Year of Service n	Running Cost (Rs.) R(n)	Cumulative Running Cost (Rs.) Σ R(n)	Resale Value (Rs.) S	Depreciation Cost (Rs.) C - S	Total Cost (Rs.) TC	Average Cost (Rs.) ATC_n
1	2	3	4	5 = 60, 000	6 = 3 + 5	7 = 6/1
1	18, 000	18, 000	42, 000	18, 000	36, 000	36, 000.00
2	20, 270	38, 270	30, 000	30, 000	68, 270	34, 135.00
3	22, 880	61, 150	20, 400	39, 600	1, 00, 750	33, 583.00
4	26, 700	87, 850	14, 400	45, 600	1, 33, 450	33, 362.00
5	31, 800	1, 19, 650	9, 650	50, 350	1, 70, 000	34, 000.00

The calculations in table 1 reveal that the average cost is lowest during the fourth year. Hence, the machine should be replaced after every fourth year, otherwise the average cost per year for running the machine would start increasing.

REPLACEMENT OF ITEMS THAT FAIL COMPLETELY

It is somehow difficult to predict that a particular equipment will fail at a particular time. This uncertainty can be avoided by deriving the probability distribution of failures. Here, it is assumed that the failures occur only at the end of the period, say t. Thus, the objective becomes to find the value of t which minimises the total cost involved for the replacement.

Mortality Tables: These tables are used to derive the probability distribution of life span of an equipment in question. Let

M (f) = Number of survivors at any time t

M(t - 1) = Number of survivors at any time t - 1

N = Initial number of equipments

Then the probability of failure during time period t is given by

$$P(t) = \frac{(t-1) - M(t)}{N}$$

The probability that an equipment has survived to an age (t - 1). and will fail during the interval (t - 1) to t can be defined as the conditional probability of failure. It is given by

$$P_c(t) = \frac{(t-1) - M(t)}{(t-1)}$$

The probability of survival to an age t is given by

$P_s(t) = \frac{M(t)}{N}$

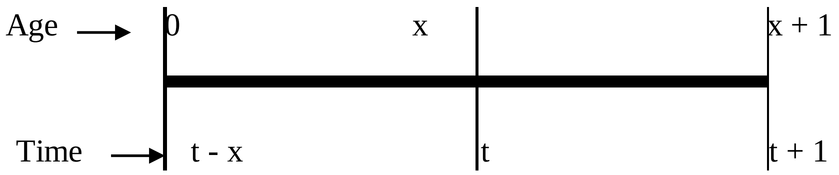
Mortality Theorem 1: A large population is subject to a given mortality law for a very long period of time. All deaths are immediately replaced by births and there are no other entries or exits. Show that the age distribution ultimately becomes stable and that the number of deaths per unit time becomes constant and is equal to the size of the total population divided by the mean age at death.

Proof: Without any loss of generality, it is assumed that death (or failure) occurs just before the age of (k + 1) years, where k is an integer. That is, life span of an item lies between t = 0 and t = k. Let us define f(t) = number of births (replacements) at time t, and

p(x) = probability of death (failure) just before the age x + 1, i.e. failure at time x.

and $\sum_{x=0}^{k} (x) = 1$

If f(t - x) represents the number of births at time t - x, t = k, k +1, k + 2,....then the age of newly borns attain the age x at time t illustrated in the figure below:



Hence the expected number of deaths of such newly borns before reaching the time t + 1 (i.e. at time t) will be

Expected number of death = $\sum_{x=0}^{k} p(x)$f(t - x), t = k, k + 1, ...

Since all deaths (failures) at time t are replaced immediately by births (replacements) at time t + 1, expected number of births are:

$$f(t + 1) = \sum_{x=0}^{k} p(x) f(t - x), \quad t = k, k + 1, \ldots.. \qquad (1)$$

The solution to the difference Eq. (9) in t can be obtained by putting the value f(t) = $A\alpha'$, where A is some constant. Then Eq. (9) becomes

$$A\alpha^{t+1} = A \sum_{x=0}^{k} (x)\, \alpha^{t-x} \qquad (2)$$

Dividing both sides of Eq. (10) by $A\alpha^{t-k}$, we get

$$\alpha^{k+1} \quad \sum_{x=0}^{k} (x)\, \alpha^{k-x} = \alpha^{k} \sum_{x=0}^{k} (x)\, \alpha^{-x}$$

$$= \alpha^{k} \{p(0) + p(1)\, \alpha^{-1} + p(2)\, \alpha^{-2} + \}$$

$$\text{or } \alpha^{k+1} - \{p(0)\, \alpha^{k} + p(1)\, \alpha^{k-1} + p(2)\, \alpha^{k-2}\} = 0 \qquad (3)$$

Equation (3) is of degree (k + 1) and will, therefore, have exactly (k + 1) roots. Let us denote the roots of Eq. (3) by $\alpha_0, \alpha_1, \alpha_2, , \alpha_k.$

For α = 1, the L. H. S of equation (3), becomes

$$\text{L.H.S} = 1 - \{p(0) + p(1) + p(2) + + p(w)\}$$

$$= 1 - \sum_{x=0}^{w} (x) = 0 \text{ R. H.S}$$

Hence, one root of Eq. (3) is α = 1. Let us denote this root by α_0. The general solution of Eq. (3) will then be of the form

$$f(t) = A_0\, \alpha^{t}_0 + A_1 \alpha^{t}_1 + \ldots\ldots + A_k\, \alpha^{t}_k$$

$$= A_0 + A_1\, \alpha^{t}_1 + \ldots\ldots + A_k\, \alpha^{t}_k$$

where $A_0, A_1, A_2, \ldots. A_k$ are constant whose values are to be calculated. (4)

Since one of the roots of Eq. (3), $\alpha_0 = 1$ is positive, according to the Descrae's sign rule all other roots $\alpha_1, \alpha_2, , \alpha_k$ will be negative and their absolute value is less than unity, i.e. $|\alpha_i| < 1$, i = 1, 2, 3, ..., k. It follows that the value of these roots tends to be zero as $\rightarrow \infty$. With the result that Eq. (4) becomes f(t) = A_0. This indicates that the number of deaths (as well as births) becomes constant at any time.

Now the problem is to determine the value of the constant A_0. For this we can proceed as follows:

Let us define

$$g(x) = \text{Probability of survival for more than } x \text{ years.}$$

$$\text{or } g(x) = 1 - \text{Prob (survivor will die before attaining the age x)}$$

$$= 1 - \{p(0) + p(1) + \dots p(x-1)\}$$

Obviously, it can be assumed that g(0) = 1.

Since the number of births as well as deaths has become constant and equal to A_0, expected number of survivors of age x is given by $A_0 . g(x)$.

As deaths are immediately replaced by births, size N of the population remains constant. That is,

$$N = A_0 \sum_{x=0}^{k} (x)\ \alpha_1, \alpha_2, \dots\dots\dots, \alpha_k$$

Or

$$A_0 = \frac{N}{\sum_{x=0}^{k} g(x)} \quad \dots\dots\dots\dots \quad (4)$$

The denominator in Eq. (4) represents the average age at death. This can also be proved as follows:

From finite differences, we know that

$$\Delta(x) = (x + 1) - x = 1$$

$$\sum_{x=a}^{b} (x)\ \Delta h(x) = f(b + 1)\ h(b + 1) - f(a)\ h(a) - \sum^{b}{}_{x=a} (x + 1)\ \Delta g(x)$$

$$= g(k + 1)\ (k + 1) - g(0)\ .\ 0 - \sum_{x=0}^{k} (x + 1)\ \Delta g(x)$$

$$= g(k + 1)\ (k + 1) - \sum_{x=0}^{k} (x + 1)\ \Delta g(x) \dots\dots\dots\dots\dots (5)$$

$$\text{But } g(k + 1) = 1 - \{p(0) + p(1) + p(2) + \dots\dots + p(k)\} = 0$$

since no one can survive for more than (k + 1) years of age and

$$\Delta g(x) = g(x + 1) - g(x)$$

$$= \{1 - p(0) - p(1) - \dots\dots - p(x)\} - \{1 - p(0) - p(1) \dots\dots p(x - 1)\} = - p(x)$$

Substituting the value of g(k + 1) and $\Delta g(x)$ in Eq. (5), we get

$$\sum_{x=0}^{k} g(x) = \sum_{x=0}^{k} (x + 1) p(x) = \text{Mean age at death}$$

Hence from Eq. (4), we get

$$A_0 = \frac{N}{Average\ age\ at\ death}.$$

STAFFING PROBLEM

The principles of replacement may be applied to formulate some useful recruitment and promotion policies for the staff working in the organisation. For this, we assume that life distribution for the services of staff in the organisation is already known.

Example 1:

An airline requires 200 assistant hostesses, 300 hostesses and 50 supervisors. Women are recruited at the age of 21, and if still in service retire at 60. Given the following life table, determine

(a) How many women should be recruited in each year?

(b) At what age should promotion take place?

Airline Hostesses' Life Record

Age	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
No. in Servie	**1000**	**600**	**480**	**384**	**307**	**261**	**228**	**206**
Age	29	30	31	32	33	34	35	36
No. in Servie	190	181	173	167	161	155	150	146
Age	37	38	39	40	41	42	43	44
No. in Servie	141	136	131	125	119	113	106	99
Age	45	46	47	48	49	50	51	52
No. in Servie	93	87	80	73	66	59	53	46
Age	53	54	55	56	57	58	59	---
No. in Servie	39	33	27	22	18	14	11	

Solution: If 1000 women had been recruited each year for the past 39 years, then the total number of them recruited at the age of 21 and those serving upto the age of 59 is 6, 480. Total number of women recruited in the airline are: 200 + 300 + 50 = 550.

(a) Number of women to be recruited every year in order to maintain a strength of 55 hostesses 550 x (1000/6840) = 85 approx.

(b) If the assistant hostesses are promoted at the age of x, then up to age (x - 1), 200 assistant hostesses will be required. Among 550 women, 200 are assistant hostesses. Therefore, out of a strength of 1, 000 there will be: 200 x (1000/550) = 364 assistant hostesses. But, from the life table given in the question, this number is available upto the age of 24 years. Thus, the promotion of assistant hostesses is due in the 25th year.

Since out of 550 recruitments only 300 hostesses are needed, if 1, 000 girls are recruited, then only 1000 x (300/550) = 545 approx will be hostesses. Hence, total number of hostesses and assistant hostesses in a recruitment will be: 545 + 364 = 909. This means, only 1000 - 909 = 91 supervisors are required. But from life table this number is available up to the age of 46 years. Thus promotion of hostesses to supervisors will be due in 47th year.

QUESTIONS

1. In the theory of replacement models construct an equation for the cost of maintaining a system as a function of the control variable t (the number of periods between group replacements).
2. State some of the simple replacement policies and give the average cost functions for the same explaining your notations.
3. The cost of maintenance of a machine is given as a function that the average annual cost will be minimised by replacing the machine when the average cost to date becomes equal to the current maintenance cost.
4. Discuss the importance of replacement models.
5. Explain how the theory of replacement is used in the following problems.

(i) Replacement of items whose maintenance cost varies with time.

(ii) Replacement of items that fail completely.

6. Find the cost per period of individual replacement of an installation of 300 lighting bulbs, given the following

(a) cost of replacing individual bulb is Rs. 3

(b) Conditional probability of failure is given below:

Week Number	:	0	1	2	3	4
Conditional Probability of Failure	:	0	1/10	1/3	2/3	1

7. The following mortality rates have been observed for a special type of light bulbs.

Month	:	1	2	3	4	5
% failing at the end of the month	:	10	25	50	80	100

In an industrial unit there are 1,000 special type of bulbs in use, and it costs Rs.10 to replace an individual bulb which has burnt out. If all bulbs were replaced simultaneously it would cost Rs. per bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out, and to continue replacing burnt out bulbs as they fail. At what intervals of time should the manager replace all the bulbs?

8. An airline, whose staff are subject to the same survival rates as in the previous problem, currently has a staff whose ages are distributed in the following table. It is estimated that for the next two years staff requirements will increase by 10% per year. If women are to be recruited at the age of 21, how many should be recruited for the next year and at what age will promotions take place? How many should be recruited for the following year and at what age will promotions take place?

Assistant	**21**	**22**	**23**	**24**	**25**				
Age	**90**	**50**	**30**	**20**	**10**		**(Total**		
Number							**200)**		
Hostesses	26	27	28	29	30	31	32	33	34
Age	40	35	35	30	28	26	20	18	16
Number									
Age	35	36	37	38	39	40	41		
Number	12	10	8	---	8	8	6 (Total 300)		
Supervisors									
Age	42	43	44	45	46	47	48	49	50
Number	5	4	5	3	3	3	6	2	---
Age	51	52	53	54	55	56	57	58	59
Number	---	4	3	5	---	3	2	---	2 (Total 50)

INVENTORY MODELS

INTRODUCTION

Inventory is one of the most expensive and important assets of many companies, representing as much as 50% of total invested capital. Managers have long recognised that good inventory control is crucial. On one hand, a firm can try to reduce costs by reducing on-hand inventory levels. On the other hand, customers become dissatisfied when frequent inventory outages, called stock outs occur. Thus, companies must make the balance between low and high inventory levels. As you would expect, cost minimisation is the major factor in obtaining this delicate balance.

Inventory is any stored resource that is used to satisfy a current or future need. Raw materials, work-in-progress, and finished goods are examples of inventory. Inventory levels for finished goods, such as clothes dryers, are a direct function of market demand. By using this demand information, it is possible to determine how much raw materials (eg., sheet metal, paint, and electric motors in the case of clothes dryers) and work-in-processes are needed to produce the finished product.

Every organisation has some type of inventory planning and control system. A bank has methods to control its inventory of cash. A hospital has methods to control blood supplies and other important items. State and federal governments, schools, and virtually every manufacturing and production organisation are concerned with inventory planning and control. Studying how organisations control their inventory is equivalent to studying how they achieve their objectives by supplying goods and services to their customers. Inventory is the common thread that ties all the functions and departments of the organisation together.

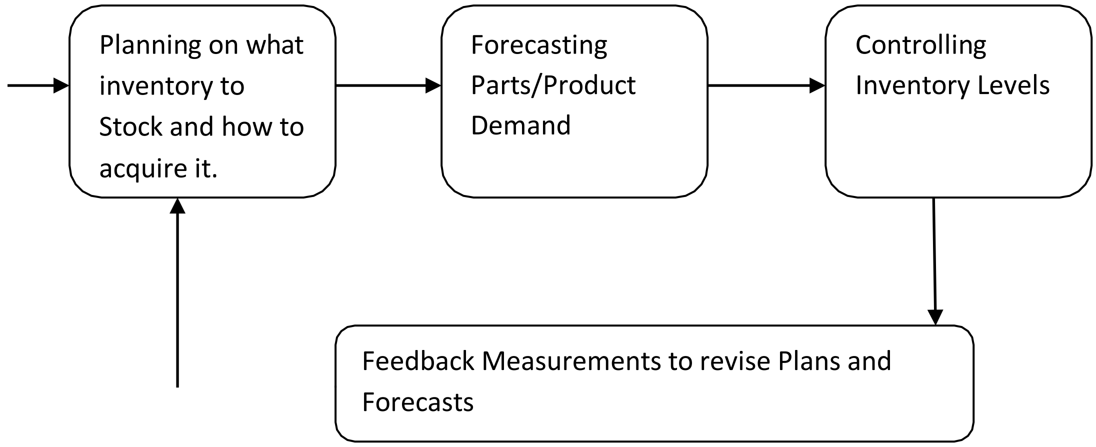
Figure 4.1 illustrates the basic components of inventory planning and control system. The planning phase involves primarily what inventory is to be stocked and how it is to be acquired (whether it is to be manufactured or purchased). This information is then used in forecasting the demand for the inventory and in controlling inventory levels. The feedback loop in figure 4.1 provides a way of revising the plan and forecast based on experiences and

observation.
Through inventory planning, an organisation determines what goods and/or services are to be produced. In cases of physical products, the organisation must also determine whether to produce these goods or to purchase them from another manufacturer. When this has been determined, the next step is to forecast the demand. Many mathematical techniques can be used in forecasting demand for a particular product. The emphasis in this chapter is on inventory control - that is, how to maintain adequate inventory levels within an organisation to support a production or procurement plan that will satisfy the forecasted demand.
In this chapter, we discuss several different inventory control models that are commonly used in practice. For each model, we provide examples of how they are analysed. Although we show the equations needed to compute the relevant parameters for each model, we use EXCEL worksheets to actually calculate these values.

IMPORTANCE OF INVENTORY CONTROL
Inventory control serves several important functions and adds a great deal of flexibility to the operation of a firm. Five main uses of inventory are as follows:
1. Decoupling function
2. Storing resources

3. Irregular supply and demand
4. Quantity discounts
5. Avoiding stockouts and shortages.

Figure 4.1 Inventory Planning and Control



1. Decoupling Function

One of the major functions of the inventory is to decouple manufacturing processes within the organisation. If a company did not store inventory, there could be many delays and inefficiencies. For example, when one manufacturing activity has to be completed before a second activity can be started, it could stop the entire process. However, stored inventory between processes could act as a buffer.

2. Storing Resources

Agricultural and seafood products often have definite seasons over which they can be harvested or caught, but the demand for these products is somewhat constant during the year. In these and similar cases, inventory could be used to store these resources.

In manufacturing process, raw materials can be stored by themselves, as work-in-process, or as finished products. Thus, if your company makes lawn movers, you may obtain lawn mower tyres from another manufacturer. If you have 400 finished lawn movers and 300 tyres in inventory, you actually have 1, 900 tyres stored in inventory. 300 tyres are stored by themselves, and 1, 600 (= 4 tyre per lawn mower X 400 lawn movers) tyres are stored on the finished lawn mowers. In the same sense, labour can be stored in inventory. If you have 500 subassemblies and it takes 50 hours of labour to produce each assembly, you actually have 25, 000 labour hours stored in inventory in the subassemblies. In general, any resource, physical or otherwise can be stored in inventory.

3. Irregular supply and demand

When the supply and demand for the inventory item is irregular, storing certain amount as inventory can be important. If the greatest demand for Diet-Delight beverage is during the summer, the Diet-Delight company will have to make sure that there is enough supply to meet this irregular demand. This might require that the company produce more of the soft drink in the winter than is actually needed in order to meet the winter demand. The inventory levels of Diet-Delight will gradually build up over the winter, but this inventory will be needed in summer. The same is true for irregular supplies.

4. Quantity Discounts

Another use of inventory is to take advantage of quantity discounts. Many suppliers offer discounts for larger orders. For example, an electric jigsaw might normally cost $10 per unit. If you order 300 or more saws at one time, your supplier may lower the cost to $8.75. Purchasing in larger quantities can substantially reduce the cost of products. There are, however, some disadvantages of buying in larger quantities. You will have higher storage costs and higher costs due to spoilage, damaged stock, theft, insurance, and so on. Furthermore, if you invest in more inventory, you will have less cash to invest elsewhere.

5. Avoiding stockouts and shortages

Another important function in inventory is to avoid stockouts and shortages. If a company is repeatedly out of stock, customers are likely to go elsewhere to satisfy their needs. Lost goodwill can be an expensive price to pay for not having the right item at the right time.

INVENTORY CONTROL DECISIONS

Even though there are literally millions of different types of products manufactured in our society, there are only two fundamental decisions that you have to make when controlling inventory:

1. How much to order
2. When to order

Table 4.1 Inventory Cost Factors

Operating Cost Factors	Carrying Cost Factors
Developing and sending purchase orders Processing and inspecting incoming inventory Bill Paying Inventory Inquiries Utilities, phone bills, and so on for the purchasing department Salaries and wages for purchasing department employees Supplies such as forms and paper for the purchasing department	Cost of Capital Taxes Insurance Spoilage Theft Obsolescence Salaries and wages for warehouse employees Utilities and building costs for the warehouse Supplies such as forms and papers for the warehouse

The purpose of all inventory models is to determine how much to order and when to order. As you know, inventory fulfils many important functions in an organisation. But as the inventory levels go up to provide these functions, the cost of holding and storing inventories also increases. Thus, we must reach a fine balance in establishing inventory levels. A major objective in controlling inventory is to minimise total inventory costs. Some of the most significant inventory costs are as follows:

1. Cost of the items
2. Cost of ordering
3. Cost of carrying or holding inventory
4. Cost of stockouts
5. Cost of safety stock, the additional inventory that may be held to help avoid stockouts.

The inventory models discussed in the first part of this chapter assume that demand and the time it takes to receive an order are known and constant, and that no quantity discounts are given. When this is the case, the most significant costs are the cost of placing an order and the cost of holding inventory items over a period of time. Table 4.1 provides a list of important factors that make up these costs.

ECONOMIC ORDER QUANTITY: DETERMINING HOW MUCH TO ORDER

The Economic Order Quantity (EOQ) model is one of the oldest and most commonly known inventory control techniques. Research on its use dates back to a 1915 publication by Ford W. Harris. This model is still used by a large number of organisations today. This technique is relatively easy to use, but it makes a number of assumptions. Some of the more important assumptions are as follows:

ASSUMPTIONS OF EOQ

1. Demand is known and constant.
2. The lead time - that is, the time between the placement of the order and the receipt of the order - is known and constant.
3. The receipt of inventory is instantaneous. In other words, the inventory from an order arrives in one batch, at one point in time.
4. Quantity discounts are not possible.
5. The only variable costs are the cost of placing an order, ordering cost, and the cost of holding or storing inventory over time, carrying or holding cost.
6. If orders are placed at the right time, stockouts and shortages can be completely avoided.

With these assumptions, inventory usage has a sawtooth shape, as in Figure 4.2. Here, Q represents the amount that is ordered. If this amount is 500 units, all 500 units are arrived at one time when an order is received. Thus, the inventory level jumps from 0 to 500 units. In general, the inventory level increases from 0 to Q units when an order arrives.

Because demand is constant over time, inventory drops at a uniform rate over time. Another order is placed such that when the inventory level reaches 0, the new order is received and the inventory level again jumps to Q units, represented by the vertical lines. This process continues indefinitely over time.

OPTIMISING INVENTORY AT PROCTER & GAMBLE

Procter & Gamble (P&G) is a world leader in consumer products with annual sales of over $76 billion. Managing inventory in such a large and complex organisation requires making effective use of the right people, organisational structure, and tools. P&G's logistics planning personnel coordinate material flow, capacity, inventory, and logistics for the firm's extensive supply chain network, which comprises P&G-owned manufacturing facilities, 300 contract manufacturers, and 6, 900 unique product-category market combinations. Each supply chain requires effective management based on the latest available information, communication and planning tools to handle complex challenges and trade-offs on issues such as production batch sizes, order policies, replenishment timing, new product introductions and assortment management.

Though the effective use of inventory optimisation models, P&G has reduced its total inventory investment significantly. Spreadsheet based inventory models that locally optimise different portions of the supply chain drive nearly 60 % of P&G's business. For more complex supply chain networks (which drive about 30% of P&G's business), advanced multi-stage models yield additional average inventory reductions of 7%. P&G estimates that the use of these tools was instrumented in driving $1.5 billion in cash savings in 2009, while maintaining or increasing service levels.

Source: Based on I. Farasyn et. al. "Inventory Optimisation at P&G: Achieving real benefits through User Adoption of Inventory Tools," Interfaces 41, 1(January-February 2011): 66-78

ORDERING AND INVENTORY COSTS

The objective of most inventory models is to minimise the total cost. With the assumptions just given, the significant costs are the ordering costs and inventory carrying cost. All other costs, such as the cost of the inventory itself, are constant. Thus, if we minimise the sum of the ordering and carrying costs, we also minimise the total cost.

To help visualise this, Figure 4.3 graphs total cost as a function of the order quantity, Q. As the value of Q increases, the total number of orders placed per year decreases. Hence, the total ordering cost decreases. However as the value of Q increases, the carrying cost increases because the firm has to maintain larger average inventories.

The optimal order size, Q^*, is the quantity that minimises the total cost. Note in figure 12.3 that Q^* occurs at the point where the ordering cost curve and the carrying cost curve intersect. This is not by chance. With this particular type of cost function, the optimal quantity always occurs at a point where the ordering cost is equal to the carrying cost.

Figure 12.2 Inventory Usage Over time

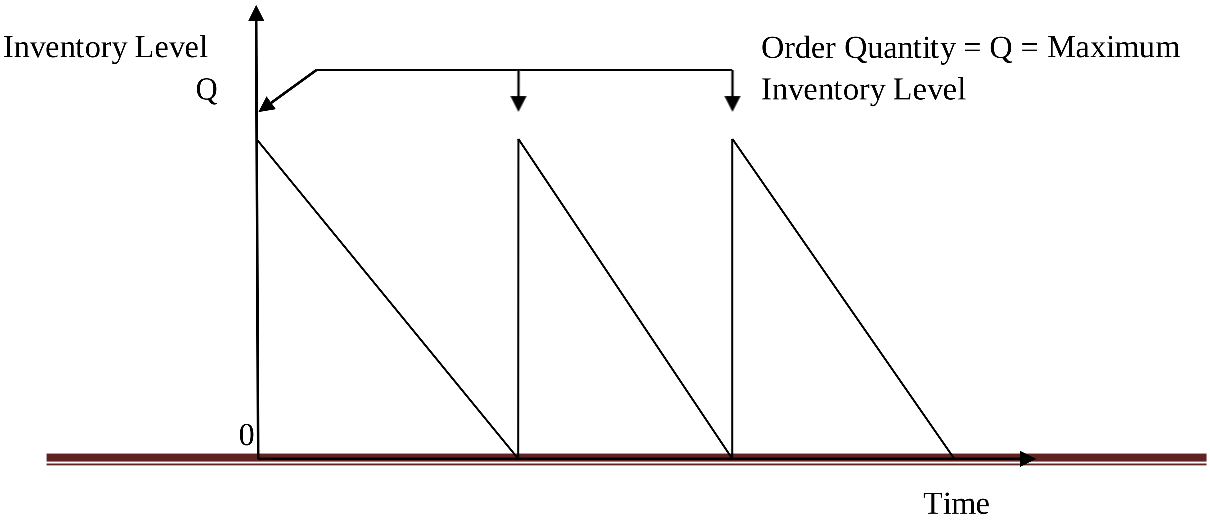
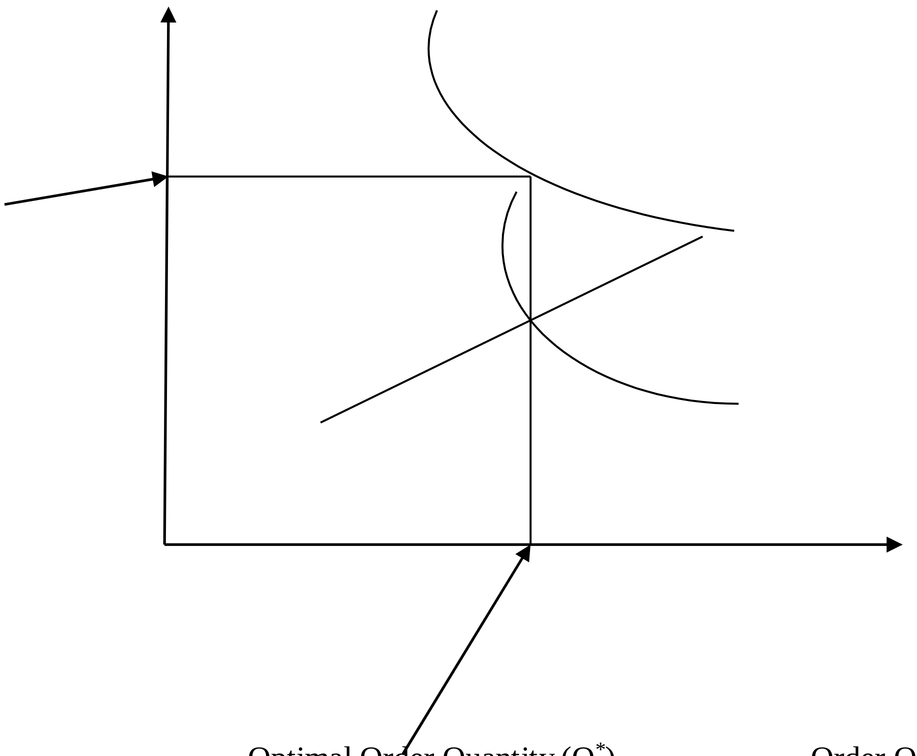


Figure 4.3 Total Cost as a Function of Order Quantity



Now that we have a better understanding of inventory costs, let us see how we can determine the value of Q^* that minimises the total cost. In determining the annual carrying cost, it is convenient to use the average inventory. Referring to figure 4.2, we see that the on-hand inventory ranges from a high of Q units to a low of zero units, with a uniform rate of decrease between these levels. Thus, the average inventory can be calculated as the average of the minimum and maximum inventory levels. That is,

$$\text{Average Inventory} = (0 + Q)/2 = Q/2 \text{-----------------------------} (1)$$

We multiply this average inventory by a factor called the annual inventory carrying cost per unit to determine the annual inventory cost.

FINDING THE ECONOMIC ORDER QUANTITY

We pointed out that the optimal order quantity, Q^*, is the point that minimises the total cost, where total cost is the sum of ordering cost and carrying cost. We also indicated graphically that the optimal order quantity was at the point where the ordering cost was equal to the carrying cost. Let us now define the following parameters:

Q^* = Optimal Order Quantity (i.e EOQ)

D = Annual Demand, in units, for the inventory item

C_o = Ordering cost per order

C_h = Carrying or holding cost per unit per year

P = Purchasing cost per unit of the inventory item

The unit carrying cost, C_h, is usually expressed in one of two ways, as follows:

1. As a fixed cost, for example, C_h is $0.50 per unit per year.

2. As a percentage (typically denoted by l) of the item's unit purchase cost or price. For example, C_h is 20% of the item's unit cost. In general

$$C_h = I \times P \text{..................................} (2)$$

For a given order quantity, Q, the ordering, holding and total costs can be computed using the following formulae:

$$\text{Total Ordering Cost} = (D/Q) \times C_o \text{ ----------------------------------- } (3)$$

$$\text{Total Carrying Cost} = (Q/2) \times C_h \text{ ------------------------------------- } (4)$$

$$\text{Total Cost} = \text{Total Ordering Cost} + \text{Total Carrying Cost} + \text{Purchase Cost}$$

$$= (D/Q) \times C_o + (Q/2) \times C_h + P \times D \text{ --------------------- } (5)$$

Observe that the total purchase cost (i.e. P x D) does not depend on the value of Q. This is so because regardless of how many orders we place each year, or how many units we order each time, we will still incur the same annual total purchase cost.

The presence of Q in the denominator for the first term makes equation (5) a non-linear equation with respect to Q. Nevertheless, because the total ordering cost is equal to the total carrying cost at the optimal value of Q, we can set the terms in equation (3) and (4) equal to each other and calculate EOQ as

$$Q^* = \sqrt{2D\, C_o / C_h} \qquad (6)$$

Inventory Control with Deterministic Models

Examples 1

1. The production department for a company requires 3, 600 kg of raw material for manufacturing a particular item per year. It has been estimated that the cost of placing an order is Rs. 36 and the cost of carrying inventory is 25% of the investment in the inventories. The price is Rs. 10/kg. The purchase manager wishes to determine an ordering policy for raw material.

Solution

From the data of the problem we know that

D = 3600 kg per year; C_o = Rs. 36 per order; C_h = 25% of the investment in inventories = 10 X 0.25 = Rs. 2.50 per kg per year.

a. The optimal lot size is given by

$Q^* = \sqrt{2D\, C_o / C_h} = \sqrt{2 \times 3600 \times 36 / 250}$ = 321.99 kg per order.

b. Optimal order cycle time

$t^* = \frac{Q^*}{D}$ = 321.99/3600 = 0.894 year

c. The minimum yearly variable inventory cost

$TVC = \sqrt{2D\, C_o C_h} = \sqrt{2 \times 3600 \times 36 \times 2.5}$ = Rs. 804.98 per year

d. The minimum yearly total inventory cost

$TC^* = TVC^* + DC$ = Rs. 804-98 + (3600 kg) (Rs. 10/kg) = Rs. 36, 804.98 per year.

2. A company operating 50 weeks in a year is concerned about its stocks of copper cable. This costs Rs. 240/- a meter and there is a demand for 8, 000 meters a week. Each replenishment costs Rs. 1, 050 for administration and Rs. 1, 650 for delivery, while holding

costs are estimated at 25% of value held a year. Assuming no shortages are allowed, what is the optimal inventory policy for the company?

How would this analysis differ if the company wanted to maximise profit rather than minimise cost? What is the gross profit if the company sell cable for Rs. 360 a meter?

Solution

From the data of the problem, we have

Demand rate (D) = 8, 000 x 50 = 4, 00, 000 metres a year

Purchase Cost (C) = Rs. 240 a unit; Ordering cost (C_o) = 1, 050 + 1, 650 = Rs. 2, 700

Holding Cost = 0.25 x 240 = Rs. 60 a meter a year

a. Optimal order quantity $Q^* = \sqrt{2D\, C_o / C_h} = \sqrt{\frac{2 \times 40{,}000 \times 2{,}700}{60}}$ = 6, 000 meters

b. Total variable inventory cost, TVC = $Q^*. C_h$ = 6, 000 x 60 = Rs. 3, 60, 000 a year

c. Total inventory cost, TC = D.C + TVC = 4, 00, 000 x 240 + 3, 60, 000 = Rs. 9, 63, 60, 000

With a turnover in excess of 96 million rupees a year inventory costs are only Rs. 3, 60, 000 or 0.36%. This figure is usually low but any well run organisation should try to make all the waivings it can, however, small.

If the company wanted to maximise profit rather than minimise cost, the analysis used would remain exactly the same. This can be demonstrated by defining selling price (SP) per unit so that gross profit per unit becomes,

Profit = Revenue - Cost

$$= D \times SP - \left(DC + \frac{D}{Q} C_o + \frac{Q}{2} C_h \right)$$

When this equation is solved to maximise profit with respect to Q as discussed earlier, we get the same result by applying usual method.

If company sells the cable for Rs. 360 a meter, its revenue is Rs. 360 x 4, 00, 000 = Rs. 14, 40, 00, 000 a year. Direct cost of Rs. 9, 63, 60, 000 is subtracted from this to get a gross profit of Rs. 4, 76, 40, 000 a year.

3. A chemical company is considering the optimal batch size for reorder of concentrated sulphuric acid. The management accountant has supplied the following information:

(1) The purchase price of H_2SO_4 is Rs. 150 per gallon.

(2) The clerical and data processing costs are Rs. 500 per order.

All the transport is done by rail. A charge of Rs. 2, 000 is made each time the special line to the factory is opened. A charge of Rs. 20 gallon is also made. The company uses 40, 000 gallons per year. Maintenance costs of stock are Rs. 400 per gallon per year.

Each gallon requires 0.5 sq ft of storage space. If warehouse space is not used, it can be rented out to another company at Rs. 200 per sq ft per annum. Available warehouse space is 1, 000 sq ft, the overhead costs being Rs. 5, 000 p.a. Assume that all free warehouse space can be rented out.

(a) Calculate the economic reorder size

(b) Calculate the minimum total annual cost of holding and reordering stock.

Solution

Based on the data of the problem, the relevant cost components which will vary over the time period due to change in lot size quantity (Q) and which will remain fixed is summarised as follows:

	Relevant Cost for EOQ Calculation	**Irrelevant Cost for EOQ Calculation**
Ordering Cost	• Clerical and data processing, Rs. 500; • Rail transport, Rs. 2000	• Rail transport, Rs. 20 per gallon because a fixed money of Rs. 40, 000 x 20 = Rs. 8, 00, 000 will incur irrespective of size of Q.
Carrying cost	• Maintenance cost, Rs. 200; • Rented cost, Rs. 200/2 = Rs. 100	• Overhead cost, Rs. 5, 000

Thus the relevant costs needed for calculating EOQ are: C_o = Rs. 2, 500; C_h = Rs. 300.

Q^* (EOQ) = $\sqrt{\frac{2DCo}{Ch}} = \sqrt{\frac{2\ x\ 40{,}000\ x\ 2{,}500}{300}}$ = Rs. 817 (approx.) gallons.

Hence, the minimum total annual costs are:

Ordering: $\frac{D}{Q*}$ C_o = $\frac{40{,}000}{817}$x 2, 500 = Rs. 1, 22, 399.02

Carrying: $\frac{Q*}{2}$ C_h = $\frac{817}{2}$ x 300 = Rs. 1, 22, 250.00

Total = Rs. 2, 44, 949.02

Rail transport: 40, 000 x 2 = Rs. 8, 00, 000

Storage Overhead cost: = Rs. 5, 000

Purchase Cost: 40, 000 x 150 = Rs. 60, 00, 000

Total = Rs. 70, 49, 949.02

4. A contractor has to supply 10, 000 bearings per day to an automobile manufacturer. He finds that when he starts production run, he can produce 25, 000 bearings per day. The cost of holding a bearing in stock for a year is Rs. 2 and the set-up cost of a production run is Rs. 180. How frequently should production run be made?

Solution

From the data of the problem in usual notation, we have

C_o = Rs. 1, 800 per production run; C_h = Rs. 2 per year

p = Rs. 25, 000 bearings per day d = 10, 000 bearings per day

D = 10, 000 x 300 = 30, 00, 000 units/year (assuming 300 working days in the year).

a. Economic batch quantity for each production run is given by

$Q^* = \sqrt{\frac{2DCo}{Ch}}\left(\frac{}{p\text{-}d}\right) = \sqrt{\frac{2\ x\ 30{,}00{,}000\ x\ 1{,}800}{2}}\left(\frac{25{,}000}{25{,}000-10{,}000}\right)$ = 1, 04, 446 bearings

b. Frequency of production cycles

$t^* = \frac{Q*}{d}$ = 1, 04, 446/10, 000 = 10.44 days.

5. A commodity is to be supplied at a constant rate of 25 units per day. A penalty cost is being charged at the rate of Rs. 10 per unit per day late for missing the scheduled delivery date. The cost of carrying the commodity in inventory is Rs. 16 per unit per month. The production process is such that each month (30 days) a batch of items is started and are available for delivery any time after the end of the month. Find the optimal level of inventory at the beginning of each month.

Solution:

From the data in usual notations, we have

D = 25 units / day; C_h = 16/30 = 0.53 per unit per day; C_o = Rs. 10 per unit per day;

t = 30 days

Thus optimal inventory level is given by

$M^* = [\frac{10}{0.53+10}]$ (25) (30) = 712 units.

SELECTIVE INVENTORY CONTROL TECHNIQUES

In practice when a firm maintained large number of items in its inventory, obviously all items cannot, and need not be controlled (i.e. keeping record of time interval between successive reviews of demand; order frequencies; expected demand rate; order quantities etc.) with equal attention. All items are not of equal importance to the firm in such terms as sales, profits, availability etc. One way of exercising proper degree of control overall and various types of items held in stocks is to classify them into groups (or classes) on the basis of the degree of control or intensity of management effort that they require.

By selectively applying inventory control policies to these different groups, inventory objectives can be achieved with lower inventory levels than with a single policy applied to all items. These techniques are also known as selective multi-item inventory control techniques.

In this section, we shall consider certain group classifications such as: ABC, VED, FSN, HML, XYZ etc.

Classification	**Basis of Classification**	**Purpose**
ABC {Always, Better, Control}	Value of consumption	To control raw material, components and work-in-progress inventories in the normal course of business.
HML {High, Medium, Low}	Unit price of the material	Mainly to control purchases
XYZ	Value of items in storage	To review the inventories and their uses at scheduled intervals.
VED {Vital, Essential, Desirable}	Criticality of the component	To determine the stocking levels of spare parts.
FSN {Fast, Slow, Non-moving}	Consumption pattern of the component	To control obsolescence

ABC ANALYSIS

The ABC analysis consists of separating the inventory items into three groups: A, B and C according to their annual cost volume consumption (unit cost x annual consumption). Although the break points between these groups vary according to individual business conditions, a common break down might be as follows:

Category or Group	**Percentage of the item**	**% of the Total Annual Value of the inventories (Rs.)**
A	10 - 20	70 - 85
B	20 - 30	10 - 25
C	60 - 70	5 - 15

This type of classification is also known as the principle of law of Vital Few and Trivial Many. The ABC analysis facilitates analysis of yearly consumption value of items in the store to identify the vital few items which are generally referred to as A category items. Generally, these items accounting for about 70% of the total money value of consumption. Items accounting for about 25% of the total money value of consumption are called B category items and the remaining ones accounting for about 5% consumption value as C category items.

Carrying on the ABC analysis of the store items helps identifying the few items that are vital from financial point of view and require careful watch, scrutiny and follow-up. The application of ABC analysis extends overall of the aspects of materials management like purchasing, inventory control, value analysis etc.

After the items are so classified, the inventory control policies are made on the basis of the classification 'A' category items require special managerial attention, therefore, fixed-interval inventory control system might be used for these items. 'C' category items can be managed in a little casual manner. For these items, a fixed-order quantity system might be used. The order quantities can be relatively large without incurring excessive costs. A large reserve stock can also be maintained. 'B' items are not so costly as to require special managerial attention, but these are not so cheap as to ignore overstocking, therefore (s, S) inventory control system might be used for these items.

The procedure of ABC analysis is summarised in the following steps:

Step 1

Obtain data on the annual usage (or consumption) in units and unit cost of each inventory unit. Multiple the annual usage in units and the value of each item to get annual value for each of these items.

Annual Value = Unit Cost x Annual Consumption

Step 2

Arrange these inventory items in a decreasing order of their value computed in step 1.

Step 3

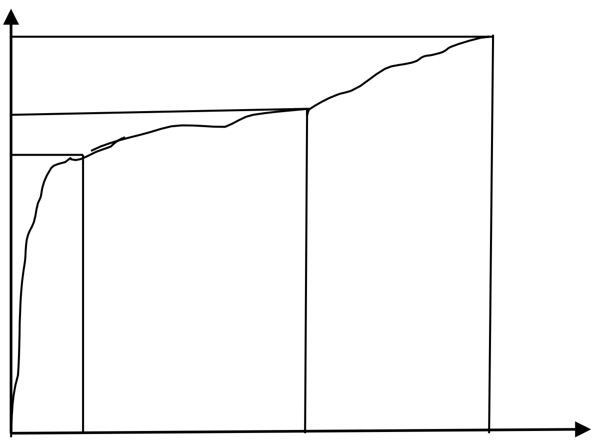
Express the annual value of each item as percentage of the total value of all items. Also compute the cumulative percentage of annual consumption rupees spent.

Step 4

Obtain the percentage value for each of the items. That is, if there are 50 items involved in classification, then each item would represent 100/50 = 2 percent of the total items. Also cumulate these percentage values.

Step 5

Draw a graph between cumulative percentage of items (on x-axis) and cumulative annual percentage of usage value (on y-axis), and mark cut-off points where the graph changes slope as shown in figure.



Example 1

A company produces a mix of high technology products for use in hospitals. The annual sales data are as follows:

Product Type	Number of Units	Unit Price (Rs.)	Product Type	Number of Units	Unit Price (Rs.)
1	1, 000	2.50	10	600	1.62
2	250	0.55	11	25	33.00
3	150	6.50	12	4	15.50

4	300	1.00	13	1, 000	5.00
5	100	1.50	14	2, 850	2.50
6	700	1.43	15	10	0.83
7	500	7.00	16	355	0.98
8	15	4.98	17	50	1.37
9	1, 000	0.75	18	393	1.85

For inventory control reasons, the company wants to classify these items into three groups A, B and C on the basis of annual sales value of each item. You please help the company.

Solution

The annual sales volume (in Rs.) for each product and the item ranking on the basis of this volume is shown in Table.

Product ranking as per sales volume

Product Type	**Number of Units**	**Unit Price (Rs.)**	**Annual Sales Volume (Rs.)**	**Ranking**
1	1000	2.50	2, 500.00	4
2	250	0.55	137.50	14
3	150	6.50	975.00	6
4	300	1.00	300.00	12
5	100	1.50	150.00	13
6	700	1.43	1001.00	5
7	500	7.00	3500.00	3
8	15	4.98	74.70	16
9	1000	0.75	750.00	9
10	600	1.62	972.00	7
11	25	33.00	825.00	8
12	4	15.50	77.50	15
13	1000	5.00	5000.00	2
14	2850	2.50	7125.00	1
15	10	0.83	8.30	18
16	355	0.98	347.90	11
17	40	1.37	54.80	17
18	393	1.85	727.05	10

The cumulative percentage of products and cumulative percentage of sales for each product is given in table for the purpose of ABC classification:

ABC Classification

Rank	**Product**	**Product Cumulative % of products**	**Annual Sales Volume (Rs.)**	**Cumulative Annual Sales Volume (Rs.)**	**Cumulative % Product Class of Sales**
1	14	5.56	7, 125.00	7,125.00	29.05 A product 11.11%
2	13	11.11	5, 000.00	12,125.00	49.43 products and 49.43 Rs.
3	7	16.67	3, 500.00	15,625.00	63.70 B products: 38.89%

4	1	22.22	2, 500.00	18,125.00	73.90 Products and 42.91 Rs.
5	6	27.78	1, 001.00	19,125.00	77.97
6	3	33.33	975.00	20,101.00	81.95
7	10	38.89	825.00	21,073.00	85.92
8	11	44.44	750.00	21,898.00	89.28
9	9	50.00	727.05	22,648.00	92.34
10	18	55.56	347.90	23,375.00	95.30 C Products 44.44%
11	16	61.11	300.00	23,722.00	96.72 products and 7.66 Rs.
12	4	66.67	150.00	24,022.00	97.94
13	5	72.22	137.50	24,172.95	98.56
14	2	77.78	27.50	24,310.45	99.12
15	12	83.33	24.70	24,387.95	99.43
16	8	88.89	54.80	24,462.65	99.74
17	17	94.44	8.30	24,517.45	99.96
18	15	100.00		24,525.75	100.00

The percentage of products and percentage of annual sales volume can also be plotted on the graph.

VED Analysis

This analysis helps in separating the inventory items into three groups according to their criticality, usually called V, E, and D items in that order, VED classification calls for classification of items as Vital, Essential and Desirable.

V items are considered vital for smooth running of the system and without these items the whole system becomes inoperative. Thus adequate stock is required all the time.

E items are considered essential to the efficient running of the system and non-availability of these items reduces the efficiency of the system.

D items neither stop the system nor reduce its efficiency, but availability of such items will lead to increase in efficiency and reduction of failure.

This classification is largely useful in controlling inventory of spare parts. It can also be used in case of such raw materials whose availability is rare.

ABC analysis and VED analysis can also be combined to control the stocking of spare parts based on the desired customer service level as shown in the table.

ABC Classification	**VED Classification**		
	V	**E**	**D**
A	Constant control, Regular follow-up Low stocks and ordering more frequently	Average stock; No risk of stock outs.	No stock
B	Average stocks	Average stock; some	Very low stocks;

	No risk of stock outs	risk can be taken	Some risk can be taken.
C	High Stocks Restricted orders; No risk	Average Stock; some risk can be taken.	Low stocks; some risk can be taken.

HML Analysis

Based on the unit price of items, the HML classification separates inventory items, as High price, Medium price and Low price. This analysis is helpful to control purchase of various items for inventory.

FSN Analysis

The consumption pattern or inventory items forms the basis for FSN analysis. Items are classified as Fast-moving, Slow-moving and Non-moving. Sometimes items are also classified as FSND: Fast-moving; Slow-moving; Normal-moving and Dead (or Non-moving). This classification is based on the movement (or consumption pattern) and therefore helps to controlling obsolescence of various items by determining the distribution and handling patterns. Cut-off points of the three classes are usually in terms of number of issues in the previous few years.

XYZ Analysis

This classification is based on the closing value of items in storage. Items whose inventory values are high and moderate are classified as X-items and Y-items respectively, while items with low inventory value are termed as Z-items.

This analysis is usually undertaken once a year during the annual stock taking exercise. This helps in identifying the items which are being stocked extensively.

This classification can also be combined with ABC classification of items to control inventory of items as shown in the table below:

ABC Classification	**XYZ Classification**		
	X	**Y**	**Z**
A	Attempt to reduce stocks	Attempt to convert Z-items	Items are within control
B	Stock and consumption is reviewed more often	Items are within control	Review stock at least twice a year
C	Dispose off the surplus items	Check and maintain the control	Review stock annually

XYZ-FSN classification exercise helps in the timely prevention of obsolescence.

XYZ Classification	**FSN Classification**		
	F	**S**	**N**
X	Light inventory control	Reduce stock to very low level	Quick disposal of items at optimum price
Y	Normal inventory control	Low level of stocks	Should be disposed as early as possible.
Z	Can reduce clerical work	Low level of stocks	Can afford to dispose

	by increasing stocks.		at lower prices.

Exercises

Questions

1. Why is inventory an important consideration for managers?
2. State the purpose of inventory control.
3. Why would not a company all the time store large quantities of inventory to avoid shortages and stockouts?
4. Describe the major decisions that must be made in inventory control.
5. State some of the assumptions made in EOQ.
6. Discuss in detail the major inventory costs that are used in EOQ.
7. Classify the following 14 items in ABC categories.

Code Number	Monthly Consumption (Rs.)	Code Number	Monthly Consumption (Rs.)
D-179-0	451	D-196	214
D-195-0	1,052	D-198-0	188
D-186-0	205	D-199	172
D-191	893	D-200	170
D-192	843	D-204	5,056
D-193	727	D-205	159
D-195	412	D-212	3,424

How the policies with regard to safety stocks, order quantity, material control and inventory system will be different for the items classified as A, B and C?

8. The following information is provided for an item:

Annual demand: 12, 000 units; Ordering Cost: Rs. 60/order
Carrying Cost: 10%; Unit cost of item: Rs. 10 and Lead Time: 10 days.

There are 300 working days in a year. Determine EOQ and number of orders per year. In the past two years the demand rate has gone as high as 70 units/day. For a reordering system based on the inventory level, what should be the buffer stock? What should be the reorder level at this buffer stock? What would be the carrying costs for a year?

9. The demand for an item in a company is 18,000 units per year, and the company can produce the item at a rate of 3,000 per month. The cost of one set-up is Rs. 500 and the holding cost of one unit per month is 15 paise. The shortage cost of one unit is Rs. 240 per year. Determine the optimum manufacturing quantity and the number of shortages. Also determine the manufacturing time and the time between set-ups.

10. A product is sold at the rate of 50 pieces per day and is manufactured at a rate of 250 pieces per day. The set-up cost of the machines is Rs.1, 000 and the storage cost is found to be Re 0.0015 per piece per day. With labour charges of Rs. 3.20 per piece, material cost at Rs.2.10 per piece and overhead cost of Rs.4.10 per piece, find the minimum cost batch size if the interest charges are 8% (assume 300 working days in a year). Compute the optimal number of cycles required in a year for the manufacture of this product.

REFERENCE: http://mu.ac.in/wp-content/uploads/2017/10/dormsem1optimizationmodel1.pdf